

Long-time behavior for a class of Feller processes

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Abstract

In this paper, as a main result, we derive a Chung-Fuchs type condition for the recurrence of Feller processes associated with pseudo-differential operators. In the Lévy process case, this condition reduces to the classical and well-known Chung-Fuchs condition. Further, we also discuss the recurrence and transience of Feller processes with respect to the dimension of the state space and Pruitt indices and the recurrence and transience of Feller-Dynkin diffusions and stable-like processes. Finally, in the one-dimensional symmetric case, we study perturbations of Feller processes which do not affect their recurrence and transience properties, and we derive sufficient conditions for their recurrence and transience in terms of the corresponding Lévy measure. In addition, some comparison conditions for recurrence and transience also in terms of the Lévy measures are obtained.

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1 Introduction

In this paper, we study the recurrence and transience property of Feller processes associated with pseudo-differential operators in terms of the symbol. To be more precise, let $(\Omega, \mathcal{F}, \{\mathbb{P}^x\}_{x \in \mathbb{R}^d}, \{\mathcal{F}_t\}_{t \geq 0}, \{\theta_t\}_{t \geq 0}, \{M_t\}_{t \geq 0})$, $\{M_t\}_{t \geq 0}$ in the sequel, be a Markov process with state space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, where $d \geq 1$ and $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ -algebra on \mathbb{R}^d . A family of linear operators $\{P_t\}_{t \geq 0}$ on $B_b(\mathbb{R}^d)$ (the space of bounded and Borel measurable functions), defined by

$$P_t f(x) := \mathbb{E}^x[f(M_t)], \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad f \in B_b(\mathbb{R}^d),$$

is associated with the process $\{M_t\}_{t \geq 0}$. Since $\{M_t\}_{t \geq 0}$ is a Markov process, the family $\{P_t\}_{t \geq 0}$ forms a *semigroup* of linear operators on the Banach space $(B_b(\mathbb{R}^d), \|\cdot\|_\infty)$, that is, $P_s \circ P_t = P_{s+t}$ and $P_0 = I$ for all $s, t \geq 0$. Here, $\|\cdot\|_\infty$ denotes the supremum norm on the space $B_b(\mathbb{R}^d)$. Moreover, the semigroup $\{P_t\}_{t \geq 0}$ is *contractive*, that is, $\|P_t f\|_\infty \leq \|f\|_\infty$ for all $t \geq 0$ and all $f \in B_b(\mathbb{R}^d)$, and *positivity preserving*, that is, $P_t f \geq 0$ for all $t \geq 0$ and all $f \in B_b(\mathbb{R}^d)$ satisfying $f \geq 0$. The

infinitesimal generator $(\mathcal{A}, \mathcal{D}_{\mathcal{A}})$ of the semigroup $\{P_t\}_{t \geq 0}$ (or of the process $\{M_t\}_{t \geq 0}$) is a linear operator $\mathcal{A} : \mathcal{D}_{\mathcal{A}} \rightarrow B_b(\mathbb{R}^d)$ defined by

$$\mathcal{A}f := \lim_{t \rightarrow 0} \frac{P_t f - f}{t}, \quad f \in \mathcal{D}_{\mathcal{A}} := \left\{ f \in B_b(\mathbb{R}^d) : \lim_{t \rightarrow 0} \frac{P_t f - f}{t} \text{ exists in } \|\cdot\|_{\infty} \right\}.$$

A Markov process $\{M_t\}_{t \geq 0}$ is said to be a *Feller process* if its corresponding semigroup $\{P_t\}_{t \geq 0}$ forms a *Feller semigroup*. This means that the family $\{P_t\}_{t \geq 0}$ is a semigroup of linear operators on the Banach space $(C_{\infty}(\mathbb{R}^d), \|\cdot\|_{\infty})$ and it is *strongly continuous*, that is,

$$\lim_{t \rightarrow 0} \|P_t f - f\|_{\infty} = 0, \quad f \in C_{\infty}(\mathbb{R}^d).$$

Here, $C_{\infty}(\mathbb{R}^d)$ denotes the space of continuous functions vanishing at infinity. Let us remark that every Feller process possesses the strong Markov property and has càdlàg paths (see [10, Theorems 3.4.19 and 3.5.14]). In the case of Feller processes, we call $(\mathcal{A}, \mathcal{D}_{\mathcal{A}})$ the *Feller generator* for short. Note that, in this case, $\mathcal{D}_{\mathcal{A}} \subseteq C_{\infty}(\mathbb{R}^d)$ and $\mathcal{A}(\mathcal{D}_{\mathcal{A}}) \subseteq C_{\infty}(\mathbb{R}^d)$. Further, if the set of smooth functions with compact support $C_c^{\infty}(\mathbb{R}^d)$ is contained in $\mathcal{D}_{\mathcal{A}}$, that is, if the Feller generator $(\mathcal{A}, \mathcal{D}_{\mathcal{A}})$ of the Feller process $\{M_t\}_{t \geq 0}$ satisfies

$$(C1) \quad C_c^{\infty}(\mathbb{R}^d) \subseteq \mathcal{D}_{\mathcal{A}},$$

then, according to [4, Theorem 3.4], $\mathcal{A}|_{C_c^{\infty}(\mathbb{R}^d)}$ is a *pseudo-differential operator*, that is, it can be written in the form

$$\mathcal{A}|_{C_c^{\infty}(\mathbb{R}^d)} f(x) = - \int_{\mathbb{R}^d} q(x, \xi) e^{i\langle \xi, x \rangle} \mathcal{F}(f)(\xi) d\xi,$$

where $\mathcal{F}(f)(\xi) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle \xi, x \rangle} f(x) dx$ denotes the Fourier transform of the function $f(x)$. The function $q : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$ is called the *symbol* of the pseudo-differential operator. It is measurable and locally bounded in (x, ξ) and continuous and negative definite as a function of ξ . Hence, by [9, Theorem 3.7.7], the function $\xi \mapsto q(x, \xi)$ has for each $x \in \mathbb{R}^d$ the following Lévy-Khintchine representation

$$q(x, \xi) = a(x) - i\langle \xi, b(x) \rangle + \frac{1}{2} \langle \xi, c(x) \xi \rangle - \int_{\mathbb{R}^d} \left(e^{i\langle \xi, y \rangle} - 1 - i\langle \xi, y \rangle 1_{\{|z| \leq 1\}}(y) \right) \nu(x, dy),$$

where $a(x)$ is a nonnegative Borel measurable function, $b(x)$ is an \mathbb{R}^d -valued Borel measurable function, $c(x) := (c_{ij}(x))_{1 \leq i, j \leq d}$ is a symmetric nonnegative definite $d \times d$ matrix-valued Borel measurable function and $\nu(x, dy)$ is a Borel kernel on $\mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d)$, called the *Lévy measure*, satisfying

$$\nu(x, \{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} \min\{1, |y|^2\} \nu(x, dy) < \infty, \quad x \in \mathbb{R}^d.$$

The quadruple $(a(x), b(x), c(x), \nu(x, dy))$ is called the *Lévy-quadruple* of the pseudo-differential operator $\mathcal{A}|_{C_c^{\infty}(\mathbb{R}^d)}$ (or of the symbol $q(x, \xi)$). In the sequel, we assume the following conditions on the symbol $q(x, \xi)$:

$$(C2) \quad \|q(\cdot, \xi)\|_{\infty} \leq c(1 + |\xi|^2) \text{ for some } c \geq 0 \text{ and all } \xi \in \mathbb{R}^d$$

$$(C3) \quad q(x, 0) = a(x) = 0 \text{ for all } x \in \mathbb{R}^d.$$

Let us remark that, according to [25, Lemma 2.1], condition **(C2)** is equivalent with the boundedness of the coefficients of the symbol $q(x, \xi)$, that is,

$$\|a\|_\infty + \|b\|_\infty + \|c\|_\infty + \left\| \int_{\mathbb{R}^d} \min\{1, y^2\} \nu(\cdot, dy) \right\|_\infty < \infty,$$

and, according to [24, Theorem 5.2], condition **(C3)** (together with condition **(C2)**) is equivalent with the conservativeness property of the process $\{M_t\}_{t \geq 0}$, that is, $\mathbb{P}^x(M_t \in \mathbb{R}^d) = 1$ for all $t \geq 0$ and all $x \in \mathbb{R}^d$. In the case when the symbol $q(x, \xi)$ does not depend on the variable $x \in \mathbb{R}^d$, $\{M_t\}_{t \geq 0}$ becomes a *Lévy process*, that is, a stochastic process with stationary and independent increments and càdlàg paths. Moreover, unlike Feller processes, every Lévy process is uniquely and completely characterized through its corresponding symbol (see [23, Theorems 7.10 and 8.1]). According to this, it is not hard to check that every Lévy process satisfies conditions **(C1)**-**(C3)** (see [23, Theorem 31.5]). Thus, the class of processes we consider in this paper contains the class of Lévy processes.

In this paper, our main aim is to investigate the recurrence and transience property of Feller processes satisfying conditions **(C1)**-**(C3)**. Except for Lévy processes, whose recurrence and transience property has been studied extensively in [23], a few special cases of this problem have been considered in the literature. More precisely, in [2], [6], [7], [18], [19], [20] and [21], by using different techniques (an overshoot approach, characteristics of semimartingale approach and an approach through the Foster-Lypunov drift criteria), the authors have considered the recurrence and transience of one-dimensional Feller processes determined by a symbol of the form $q(x, \xi) = \gamma(x)|\xi|^{\alpha(x)}$ (stable-like processes), where $\alpha : \mathbb{R} \rightarrow (0, 2)$ and $\gamma : \mathbb{R} \rightarrow (0, \infty)$ (see Section 2 for the exact definition of these processes). Further, by using the Foster-Lyapunov drift criteria (see [13] or [14]), in [31] the author has derived sufficient conditions for recurrence of one-dimensional Feller processes in terms of their Lévy quadruples. Finally, in [28], by analyzing the characteristic function of Feller processes, the authors have derived a Chung-Fuchs type condition for transience (see Theorem 1.3 for details). In this paper, our goal is to extend the above mentioned results in several different aspects as well as to answer some natural questions regarding the recurrence and transience in order to better understand the long-time behavior of Feller processes. To be more precise, our main goal is to derive a Chung-Fuchs type condition for the recurrence of a Feller process. Furthermore, we study recurrence and transience in relation to the dimension of the state space and Pruitt indices and recurrence and transience of Feller-Dynkin diffusions and stable-like processes. Finally, we study perturbations of symbols which will not affect the recurrence and transience of the underlying Feller processes and we derive sufficient conditions for the recurrence and transience in terms of the underlying Lévy measure and some comparison conditions for recurrence and transience also in terms of their Lévy measures.

Before stating the main results of this paper, we recall relevant definitions of the recurrence and transience of Markov processes in the sense of S. P. Meyn and R. L. Tweedie (see [14] or [30]).

Definition 1.1. Let $\{M_t\}_{t \geq 0}$ be a strong Markov process with càdlàg paths on the state space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $d \geq 1$. The process $\{M_t\}_{t \geq 0}$ is called

- (i) *irreducible* if there exists a σ -finite measure $\varphi(dy)$ on $\mathcal{B}(\mathbb{R}^d)$ such that whenever $\varphi(B) > 0$ we have $\int_0^\infty \mathbb{P}^x(M_t \in B) dt > 0$ for all $x \in \mathbb{R}^d$.
- (ii) *recurrent* if it is φ -irreducible and if $\varphi(B) > 0$ implies $\int_0^\infty \mathbb{P}^x(M_t \in B) dt = \infty$ for all $x \in \mathbb{R}^d$.
- (iii) *Harris recurrent* if it is φ -irreducible and if $\varphi(B) > 0$ implies $\mathbb{P}^x(\tau_B < \infty) = 1$ for all $x \in \mathbb{R}^d$, where $\tau_B := \inf\{t \geq 0 : M_t \in B\}$.

(iv) transient if it is φ -irreducible and if there exists a countable covering of \mathbb{R}^d with sets $\{B_j\}_{j \in \mathbb{N}} \subseteq \mathcal{B}(\mathbb{R}^d)$, such that for each $j \in \mathbb{N}$ there is a finite constant $c_j \geq 0$ such that $\int_0^\infty \mathbb{P}^x(M_t \in B_j) dt \leq c_j$ holds for all $x \in \mathbb{R}^d$.

Let us remark that if $\{M_t\}_{t \geq 0}$ is a φ -irreducible Markov process, then the irreducibility measure $\varphi(dy)$ can be maximized, that is, there exists a unique “maximal” irreducibility measure $\psi(dy)$ such that for any measure $\bar{\varphi}(dy)$, $\{M_t\}_{t \geq 0}$ is $\bar{\varphi}$ -irreducible if, and only if, $\bar{\varphi} \ll \psi$ (see [30, Theorem 2.1]). According to this, from now on, when we refer to irreducibility measure we actually refer to the maximal irreducibility measure. In the sequel, we consider only the so-called open set irreducible Markov processes, that is, we consider only ψ -irreducible Markov processes whose maximal irreducibility measure $\psi(dy)$ satisfies the following *open set irreducibility* condition:

(C4) $\psi(O) > 0$ for every open set $O \subseteq \mathbb{R}^d$.

Obviously, the Lebesgue measure $\lambda(dy)$ satisfies condition (C4) and a Markov process $\{M_t\}_{t \geq 0}$ will be λ -irreducible if $\mathbb{P}^x(M_t \in B) > 0$ for all $t > 0$ and all $x \in \mathbb{R}^d$ whenever $\lambda(B) > 0$. In particular, the process $\{M_t\}_{t \geq 0}$ will be λ -irreducible if the transition kernel $\mathbb{P}^x(M_t \in dy)$ possesses a density function $p(t, x, y)$ such that $p(t, x, y) > 0$ for all $t > 0$ and all $x, y \in \mathbb{R}^d$. If $\{M_t\}_{t \geq 0}$ is a Feller process satisfying conditions (C1)-(C3) and, additionally, the following sector condition

$$\sup_{x \in \mathbb{R}^d} |\operatorname{Im} q(x, \xi)| \leq c \inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi) \quad (1.1)$$

for some $0 \leq c < 1$ and all $\xi \in \mathbb{R}^d$, then a sufficient condition for the existence of a density function $p(t, x, y)$, in terms of the symbol $q(x, \xi)$, has been given in [28, Theorem 1.1] as the Hartman-Wintner condition

$$\lim_{|\xi| \rightarrow \infty} \frac{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)}{\log(1 + |\xi|)} = \infty \quad (1.2)$$

(see also Theorem 2.6). Let us remark that the sector condition in (1.1) means that there is no dominating drift term. Further, it is well known that every ψ -irreducible Markov process is either recurrent or transient (see [30, Theorem 2.3]) and, clearly, every Harris recurrent Markov process is recurrent but in general, these two properties are not equivalent. They differ on the set of the irreducibility measure zero (see [30, Theorem 2.5]). However, for a Feller process satisfying conditions (C1)-(C4) these two properties are equivalent (see Proposition 2.1).

Throughout this paper, the symbol $\{F_t\}_{t \geq 0}$ denotes a Feller process satisfying conditions (C1)-(C4). Such a process is called a *nice Feller process*. We say that $\{F_t\}_{t \geq 0}$ is a *symmetric nice Feller process* if its corresponding symbol satisfies $q(x, \xi) = \operatorname{Re} q(x, \xi)$, that is, if $b(x) = 0$ and $\nu(x, dy)$ are symmetric measures for all $x \in \mathbb{R}^d$. Also, a Lévy process is denoted by $\{L_t\}_{t \geq 0}$.

The main result of this paper, the proof of which is given in Section 2, is the following Chung-Fuchs type condition for the recurrence of nice Feller processes.

Theorem 1.2. *Let $\{F_t\}_{t \geq 0}$ be a nice Feller process with symbol $q(x, \xi)$. If $\operatorname{Re} \mathbb{E}^0[e^{i\langle \xi, F_t \rangle}] \geq 0$ for all $t \geq 0$ and all $\xi \in \mathbb{R}^d$ and*

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\sup_{x \in \mathbb{R}^d} |q(x, \xi)|} = \infty \quad \text{for some } r > 0, \quad (1.3)$$

then $\{F_t\}_{t \geq 0}$ is recurrent.

The Chung-Fuchs type condition for transience of nice Feller processes has been derived in [28, Theorem 1.2] and it reads as follows.

Theorem 1.3. *Let $\{F_t\}_{t \geq 0}$ be a nice Feller process with symbol $q(x, \xi)$. If $\{F_t\}_{t \geq 0}$ satisfies the sector condition in (1.1) and*

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)} < \infty \quad \text{for some } r > 0, \quad (1.4)$$

then $\{F_t\}_{t \geq 0}$ is transient.

In the case when $\{F_t\}_{t \geq 0}$ is a Lévy process with symbol $q(\xi)$, the Lévy-Khintchine formula yields $\mathbb{E}^0[e^{i\langle \xi, L_t \rangle}] = e^{-tq(\xi)}$ for all $t \geq 0$ and all $\xi \in \mathbb{R}^d$ (see [23, Theorems 7.10 and 8.1]). In particular, if $\{F_t\}_{t \geq 0}$ is a symmetric Lévy process, then $\operatorname{Re} \mathbb{E}^0[e^{i\langle \xi, L_t \rangle}] = e^{-tq(\xi)} \geq 0$. Thus, we get the well-known Chung-Fuchs conditions (see [23, Theorem 37.5]). This shows that the conditions of Theorems 1.2 and 1.3 are sharp for Lévy processes. Clearly, for each frozen $x \in \mathbb{R}^d$, $q(x, \xi)$ is the symbol of some Lévy process $\{L_t^x\}_{t \geq 0}$. Thus, intuitively, Theorem 1.2 says that if all the Lévy processes $\{L_t^x\}_{t \geq 0}$, $x \in \mathbb{R}^d$, are recurrent, then the Feller process $\{F_t\}_{t \geq 0}$ is also recurrent. Similarly, Theorem 1.3 says that if all the Lévy processes $\{L_t^x\}_{t \geq 0}$, $x \in \mathbb{R}^d$, are transient, then the Feller process $\{F_t\}_{t \geq 0}$ is also transient.

As is well known, the fact whether or not a Lévy process is recurrent or transient depends on the dimension of the state space. Hence, it is natural to expect that a similar result holds in the situation of nice Feller processes. In Theorem 2.8, we discuss this dependence, which again generalizes the Lévy process situation (see [23, Theorems 37.8 and 37.14]). More precisely, we prove that when $d = 1, 2$ and $q(x, \xi) = \operatorname{Re} q(x, \xi)$ for all $x \in \mathbb{R}^d$, then

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |y|^2 \nu(x, dy) < \infty$$

implies (1.3), and when $d \geq 3$, then

$$\liminf_{|\xi| \rightarrow 0} \frac{\sup_{c > 0} \inf_{x \in \mathbb{R}^d} \left(\langle \xi, c(x)\xi \rangle + \int_{\{|y| \leq c\}} \langle \xi, y \rangle^2 \nu(x, dy) \right)}{|\xi|^2} > 0,$$

implies (1.4). In particular, a symmetric *Feller-Dynkin diffusion*, that is, a symmetric nice Feller process determined by a symbol of the form $q(x, \xi) = \frac{1}{2} \langle \xi, c(x)\xi \rangle$, is recurrent if, and only if, $d = 1, 2$ (see Theorem 2.9).

Recall that a *rotationally invariant stable Lévy process* $\{L_t\}_{t \geq 0}$ is a Lévy process with symbol given by $q(\xi) = \gamma |\xi|^\alpha$, where $\alpha \in (0, 2]$ and $\gamma \in (0, \infty)$. The parameters α and γ are called the stability parameter and the scaling parameter, respectively (see [23, Chapter 3] for details). Note that when $\alpha = 2$, then $\{L_t\}_{t \geq 0}$ becomes a Brownian motion. It is well known that the recurrence and transience property of $\{L_t\}_{t \geq 0}$ depends on the index of stability α . More precisely, if $d \geq 3$, then $\{L_t\}_{t \geq 0}$ is always transient, if $d = 2$, then $\{L_t\}_{t \geq 0}$ is recurrent if, and only if, $\alpha = 2$ and if $d = 1$, then $\{L_t\}_{t \geq 0}$ is recurrent if, and only if, $\alpha \geq 1$ (see [23, Theorems 37.8, 37.16 and 37.18]). The notion of stable Lévy processes has been generalized in [1]. More precisely, under some technical assumptions on the functions $\alpha : \mathbb{R}^d \rightarrow (0, 2)$ and $\gamma : \mathbb{R}^d \rightarrow (0, \infty)$ (see Section 2 for details), in [1] and [28, Theorem 3.3] it has been shown that there exists a unique nice Feller process, called a *stable-like process*, determined by a symbol of the form $q(x, \xi) = \gamma(x) |\xi|^{\alpha(x)}$. In Theorem 2.10 and Corollary 3.3, we discuss the recurrence and transience property of stable-like processes. Next, the concept of the indices of stability can be generalized to general Lévy processes through the Pruitt indices (see [15]). The Pruitt indices for nice Feller processes have been introduced in [25]. In Theorems 2.12 and 2.13, we also discuss the recurrence and transience property of nice Feller processes, as well as of Lévy processes, in terms of the Pruitt indices.

A natural problem which arises is to determine those perturbations of the symbol (or the Lévy quadruple) which will not affect the recurrence or transience property of the underlying nice Feller process. In the one-dimensional symmetric case, in Theorem 3.1, we prove that if $\{F_t^1\}_{t \geq 0}$ and $\{F_t^2\}_{t \geq 0}$ are two nice Feller process with Lévy measures $\nu_1(x, dy)$ and $\nu_2(x, dy)$, respectively, such that

$$\sup_{x \in \mathbb{R}} \int_{\mathbb{R}} y^2 |\nu_1(x, dy) - \nu_2(x, dy)| < \infty,$$

then $\{F_t^1\}_{t \geq 0}$ and $\{F_t^2\}_{t \geq 0}$ are recurrent or transient at the same time. Here, $|\mu(dy)|$ denotes the total variation measure of the signed measure $\mu(dy)$. In particular, we conclude that the recurrence and transience property of one-dimensional symmetric nice Feller processes depends only on big jumps. Further, in general it is not always easy to compute the Chung-Fuchs type conditions in (1.3) and (1.4). According to this, in the one-dimensional symmetric case, in Theorems 3.7 and 3.9, we give necessary and sufficient condition for the recurrence and transience in terms of the Lévy measure. Finally, in Theorems 3.12 and 3.13, we give some comparison conditions for the recurrence and transience in terms of the Lévy measures.

In the Lévy process case, the main ingredient in the proof of the Chung-Fuchs conditions is the fact that

$$\mathbb{E}^x[e^{i\langle \xi, L_t - x \rangle}] = e^{-tq(\xi)}$$

for all $t \geq 0$ and all $x, \xi \in \mathbb{R}^d$, where $q(\xi)$ is the symbol of the Lévy process $\{L_t\}_{t \geq 0}$ (see [23, Theorems 7.10 and 8.1]). This relation is no longer true for a general nice Feller process $\{F_t\}_{t \geq 0}$. Since $\{F_t\}_{t \geq 0}$ does not have stationary and independent increments, in particular, it is not spatially homogeneous, the characteristic function of F_t , $t \geq 0$, will now depend on the starting point $x \in \mathbb{R}^d$ and $q(x, \xi)$ is no longer the characteristic exponent of $\{F_t\}_{t \geq 0}$. Anyway, it is natural to expect that

$$\mathbb{E}^x[e^{i\langle \xi, F_t - x \rangle}] \approx e^{-tq(x, \xi)}$$

for all $t \geq 0$ and all $x, \xi \in \mathbb{R}^d$. According to this, as the main step in the proof of Theorem 1.2 we derive a lower bound for $\mathbb{E}^x[e^{i\langle \xi, F_t - x \rangle}]$. More precisely, in Lemma 2.2, we prove that for any $\varepsilon > 0$ and $\xi \in \mathbb{R}^d$ there exists $t_0 := t_0(\varepsilon, \xi) > 0$, such that for all $x \in \mathbb{R}^d$ and all $t \in [0, t_0]$ we have

$$\operatorname{Re} \mathbb{E}^x[e^{i\langle \xi, F_t - x \rangle}] \geq \exp \left[-(2 + \varepsilon)t \sup_{z \in \mathbb{R}^d} |q(z, \xi)| \right].$$

The upper bound for $\mathbb{E}^x[e^{i\langle \xi, F_t - x \rangle}]$ has been given in [28, Theorem 2.7] and it reads as follows

$$\left| \mathbb{E}^x[e^{i\langle \xi, F_t - x \rangle}] \right| \leq \exp \left[-\frac{t}{16} \inf_{z \in \mathbb{R}^d} \operatorname{Re} q(z, 2\xi) \right]$$

for all $t \geq 0$ and all $x, \xi \in \mathbb{R}^d$. The proofs of the remaining results presented in this paper are mostly based on the Chung-Fuchs type conditions in (1.3) and (1.4) and the analysis of the symbols.

The sequel of this paper is organized as follows. In Section 2, we prove Theorem 1.2 and discuss the recurrence and transience of nice Feller processes with respect to the dimension of the state space and Pruitt indices and the recurrence and transience of Feller-Dynkin diffusions and stable-like processes. Finally, in Section 3, we discuss the recurrence and transience property of one-dimensional symmetric nice Feller processes. More precisely, we study perturbations of nice Feller processes and we derive sufficient conditions for the recurrence and transience in terms of the Lévy measure and give some comparison conditions for the recurrence and transience property in terms of the Lévy measures.

2 Recurrence and transience of general nice Feller processes

We start this section with some preliminary and auxiliary results regarding the recurrence and transience property of nice Feller processes which we need in the sequel. First, recall that a semigroup $\{P_t\}_{t \geq 0}$ on $(B_b(\mathbb{R}^d), \|\cdot\|_\infty)$ is called a *C_b -Feller semigroup* if $P_t(C_b(\mathbb{R}^d)) \subseteq C_b(\mathbb{R}^d)$ for all $t \geq 0$ and it is called a *strong Feller semigroup* if $P_t(B_b(\mathbb{R}^d)) \subseteq C_b(\mathbb{R}^d)$ for all $t \geq 0$. Here, $C_b(\mathbb{R}^d)$ denotes the space of continuous and bounded functions. For sufficient conditions for a Feller semigroup to be a C_b -Feller semigroup or a strong Feller semigroup see [24] and [27].

Proposition 2.1. *Let $\{F_t\}_{t \geq 0}$ be a nice Feller process. Then the following properties are equivalent:*

- (i) $\{F_t\}_{t \geq 0}$ is recurrent
- (ii) $\{F_t\}_{t \geq 0}$ is Harris recurrent
- (iii) there exists $x \in \mathbb{R}^d$ such that

$$\mathbb{P}^x \left(\liminf_{t \rightarrow \infty} |F_t - x| = 0 \right) = 1$$

- (iv) there exists $x \in \mathbb{R}^d$ such that

$$\int_0^\infty \mathbb{P}^x(F_t \in O_x) dt = \infty$$

for all open neighborhoods $O_x \subseteq \mathbb{R}^d$ around x

- (v) there exists a compact set $C \subseteq \mathbb{R}^d$ such that

$$\mathbb{P}^x(\tau_C < \infty) = 1$$

for all $x \in \mathbb{R}^d$

- (vi) for each initial position $x \in \mathbb{R}^d$ and each covering $\{O_n\}_{n \in \mathbb{N}}$ of \mathbb{R}^d by open bounded sets we have

$$\mathbb{P}^x \left(\bigcap_{n=1}^\infty \bigcup_{m=0}^\infty \left\{ \int_m^\infty 1_{\{F_t \in O_n\}} dt = 0 \right\} \right) = 0.$$

In other words, $\{F_t\}_{t \geq 0}$ is recurrent if, and only if, for each initial position $x \in \mathbb{R}^d$ the event $\{F_t \in C^c \text{ for any compact set } C \subseteq \mathbb{R}^d \text{ and all } t \geq 0 \text{ sufficiently large}\}$ has probability 0.

In addition, if we assume that $\{F_t\}_{t \geq 0}$ is a strong Feller process, then all the statements above are also equivalent to:

- (vii) there exists a compact set $C \subseteq \mathbb{R}^d$ such that

$$\int_0^\infty \mathbb{P}^x(F_t \in C) dt = \infty$$

for all $x \in \mathbb{R}^d$

- (viii) there exist $x \in \mathbb{R}^d$ and an open bounded set $O \subseteq \mathbb{R}^d$ such that

$$\mathbb{P}^x \left(\int_0^\infty 1_{\{F_t \in O\}} dt = \infty \right) > 0.$$

Proof. First, let us remark that every Feller semigroup $\{P_t\}_{t \geq 0}$ has a unique extension onto the space $B_b(\mathbb{R}^d)$ (see [24, Section 3]). For notational simplicity, we denote this extension again by $\{P_t\}_{t \geq 0}$. Now, according to [24, Corollary 3.4 and Theorem 4.3] and [28, Lemma 2.3], $\{P_t\}_{t \geq 0}$ is a C_b -Feller semigroup.

(i) \Leftrightarrow (ii) This is an immediate consequence of [30, Theorems 4.1, 4.2 and 7.1].

(i) \Leftrightarrow (iii) This is an immediate consequence of [30, Theorem 7.1] and [2, Theorem 4.3].

(i) \Leftrightarrow (iv) This is an immediate consequence of [30, Theorems 4.1 and 7.1].

(i) \Leftrightarrow (v) This is an immediate consequence of (iii) and [12, Theorem 3.3].

(i) \Leftrightarrow (vi) This is an immediate consequence of (ii) and [30, Theorem 3.3].

(i) \Leftrightarrow (vii) By [18, Proposition 2.3], it suffices to prove that

$$\inf_{x \in C} \int_0^\infty \mathbb{P}^x(F_t \in B) e^{-t} dt > 0$$

holds for every compact set $C \subseteq \mathbb{R}^d$ and every $B \in \mathcal{B}(\mathbb{R}^d)$ satisfying $\psi(B) > 0$. Let us assume that this is not the case. Then, there exist a compact set $C \subseteq \mathbb{R}^d$, a Borel set $B \subseteq \mathbb{R}^d$ satisfying $\psi(B) > 0$ and a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq C$ with $\lim_{n \rightarrow \infty} x_n = x_0 \in C$, such that

$$\lim_{n \rightarrow \infty} \int_0^\infty \mathbb{P}^{x_n}(F_t \in B) e^{-t} dt = 0.$$

Now, by the dominated convergence theorem and the strong Feller property, it follows

$$\int_0^\infty \mathbb{P}^{x_0}(F_t \in B) e^{-t} dt = 0.$$

But this is in contradiction with the ψ -irreducibility property of $\{F_t\}_{t \geq 0}$.

(i) \Leftrightarrow (viii) This is an immediate consequence of (vii) and [18, Proposition 2.4].

□

The proof of Theorem 1.2 is based on the following lemma.

Lemma 2.2. *Let $\{F_t\}_{t \geq 0}$ be a nice Feller process with symbol $q(x, \xi)$ and let $\Phi_t(x, \xi) := \mathbb{E}^x [e^{i\langle \xi, F_t - x \rangle}]$ for $t \geq 0$ and $x, \xi \in \mathbb{R}^d$. Then, for any $\varepsilon > 0$ and $\xi \in \mathbb{R}^d$ there exists $t_0 := t_0(\varepsilon, \xi) > 0$, such that for all $x \in \mathbb{R}^d$ and all $t \in [0, t_0]$ we have*

$$\operatorname{Re} \Phi_t(x, \xi) \geq \exp \left[-(2 + \varepsilon)t \sup_{z \in \mathbb{R}^d} |q(z, \xi)| \right].$$

Proof. First, by [28, Proposition 4.2] and [25, proof of Lemma 6.3], for all $t \geq 0$ and all $x, \xi \in \mathbb{R}^d$ we have

$$\Phi_t(x, \xi) = 1 - \int_0^t P_s \left(q(\cdot, \xi) e^{i\langle \xi, \cdot - x \rangle} \right) (x) ds.$$

Recall that $\{P_t\}_{t \geq 0}$ denotes the semigroup of $\{F_t\}_{t \geq 0}$. Thus,

$$\begin{aligned} \operatorname{Re} \Phi_t(x, \xi) &= 1 - \int_0^t \int_{\mathbb{R}^d} (\cos \langle \xi, y - x \rangle \operatorname{Re} q(y, \xi) - \sin \langle \xi, y - x \rangle \operatorname{Im} q(y, \xi)) \mathbb{P}^x(F_s \in dy) ds \\ &\geq 1 - \int_0^t \int_{\mathbb{R}^d} (\operatorname{Re} q(y, \xi) + |\operatorname{Im} q(y, \xi)|) \mathbb{P}^x(F_s \in dy) ds \\ &\geq 1 - 2t \sup_{z \in \mathbb{R}^d} |q(z, \xi)|. \end{aligned}$$

Finally, for given $\varepsilon > 0$ and $\xi \in \mathbb{R}^d$, it is easy to check that for all $t \in \left[0, \frac{\ln(\frac{2+\varepsilon}{2})}{(2+\varepsilon) \sup_{z \in \mathbb{R}^d} |q(z, \xi)|}\right]$ we have

$$\operatorname{Re} \Phi_t(x, \xi) \geq 1 - 2t \sup_{z \in \mathbb{R}^d} |q(z, \xi)| \geq \exp \left[-(2 + \varepsilon)t \sup_{z \in \mathbb{R}^d} |q(z, \xi)| \right].$$

□

Now, we prove Theorem 1.2.

Proof of Theorem 1.2. First, note that, according to Proposition 2.1 (iv), it suffices to prove that

$$\mathbb{E}^0 \left[\int_0^\infty 1_{\{F_t \in O_0\}} dt \right] = \infty$$

for every open neighborhood $O_0 \subseteq \mathbb{R}^d$ around the origin. Let $a > 0$ be arbitrary. By the monotone convergence theorem, we have

$$\begin{aligned} \mathbb{E}^0 \left[\int_0^\infty 1_{\{F_t \in (-a, a)^d\}} dt \right] &= \lim_{\alpha \rightarrow 0} \mathbb{E}^0 \left[\int_0^\infty e^{-\alpha t} 1_{\{F_t \in (-a, a)^d\}} dt \right] \\ &= \lim_{\alpha \rightarrow 0} \int_0^\infty \int_{\mathbb{R}^d} e^{-\alpha t} 1_{(-a, a)^d}(y) \mathbb{P}^0(F_t \in dy) dt, \end{aligned}$$

where $(-a, a)^d := (-a, a) \times \dots \times (-a, a)$. Next, let

$$f(x) := \left(1 - \frac{|x|}{a} \right) 1_{(-a, a)}(x),$$

for $x \in \mathbb{R}$, and

$$g(x) := f(x_1) \cdots f(x_d),$$

for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Clearly, we have

$$1_{(-a, a)^d}(x) \geq g(x)$$

for all $x \in \mathbb{R}^d$. According to this, we have

$$\mathbb{E}^0 \left[\int_0^\infty 1_{\{F_t \in (-a, a)^d\}} dt \right] \geq \liminf_{\alpha \rightarrow 0} \int_0^\infty \int_{\mathbb{R}^d} e^{-\alpha t} g(y) \mathbb{P}^0(F_t \in dy) dt. \quad (2.1)$$

Further,

$$f(x) = \frac{1}{\sqrt{a}} 1_{(-\frac{a}{2}, \frac{a}{2})} * \frac{1}{\sqrt{a}} 1_{(-\frac{a}{2}, \frac{a}{2})}(x),$$

where $*$ denotes the standard convolution operator. Hence, since

$$\mathcal{F}\left(\frac{1}{\sqrt{a}}1_{(-\frac{a}{2}, \frac{a}{2})}\right)(\xi) = \frac{\sin\left(\frac{a\xi}{2}\right)}{\sqrt{a\pi}\xi},$$

we have

$$\mathcal{F}(g)(\xi) = \frac{\sin^2\left(\frac{a\xi_1}{2}\right)}{a\pi^2\xi_1^2} \dots \frac{\sin^2\left(\frac{a\xi_d}{2}\right)}{a\pi^2\xi_d^2}.$$

This and (2.1) yields

$$\begin{aligned} \mathbb{E}^0 \left[\int_0^\infty 1_{\{F_t \in (-a, a)^d\}} dt \right] &\geq \liminf_{\alpha \rightarrow 0} \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\alpha t} e^{i\langle \xi, y \rangle} \mathcal{F}(g)(\xi) d\xi \mathbb{P}^0(F_t \in dy) dt \\ &= \liminf_{\alpha \rightarrow 0} \int_0^\infty \int_{\mathbb{R}^d} e^{-\alpha t} \Phi_t(0, \xi) \mathcal{F}(g)(\xi) d\xi dt \\ &= \liminf_{\alpha \rightarrow 0} \int_0^\infty \int_{\mathbb{R}^d} e^{-\alpha t} \operatorname{Re} \Phi_t(0, \xi) \mathcal{F}(g)(\xi) d\xi dt \\ &= \liminf_{\alpha \rightarrow 0} \int_{\mathbb{R}^d} \left(\int_0^{t_0(2, \xi)} e^{-\alpha t} \operatorname{Re} \Phi_t(0, \xi) dt \right. \\ &\quad \left. + \int_{t_0(2, \xi)}^\infty e^{-\alpha t} \operatorname{Re} \Phi_t(0, \xi) dt \right) \mathcal{F}(g)(\xi) d\xi \\ &\geq \liminf_{\alpha \rightarrow 0} \int_{\mathbb{R}^d} \frac{1 - \exp\left[-\ln 2 \frac{\alpha + 4 \sup_{x \in \mathbb{R}^d} |q(x, \xi)|}{4 \sup_{x \in \mathbb{R}^d} |q(x, \xi)|}\right]}{\alpha + 4 \sup_{x \in \mathbb{R}^d} |q(x, \xi)|} \mathcal{F}(g)(\xi) d\xi, \end{aligned}$$

where in the fourth step $t_0(2, \xi) = \frac{\ln 2}{4 \sup_{x \in \mathbb{R}^d} |q(x, \xi)|}$ is given in Lemma 2.2 and in the final step we applied Lemma 2.2 and the assumption that $\operatorname{Re} \Phi_t(0, \xi) \geq 0$ for all $t \geq 0$ and all $\xi \in \mathbb{R}^d$. Now, by Fatou's lemma, we have

$$\mathbb{E}^0 \left[\int_0^\infty 1_{\{F_t \in (-a, a)^d\}} dt \right] \geq \frac{1}{8} \int_{\mathbb{R}^d} \frac{\mathcal{F}(g)(\xi)}{\sup_{x \in \mathbb{R}^d} |q(x, \xi)|} d\xi.$$

Finally, let $r > 0$ be such that

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\sup_{x \in \mathbb{R}^d} |q(x, \xi)|} = \infty.$$

Then, since

$$\lim_{a \rightarrow 0} \frac{(2\pi)^{2d}}{a^d} \mathcal{F}(g)(\xi) = 1,$$

for any $c \in (0, 1)$, all $|\xi| < r$ and all $a > 0$ small enough we have

$$\mathbb{E}^0 \left[\int_0^\infty 1_{\{F_t \in (-a, a)^d\}} dt \right] \geq \frac{ca^d}{8(2\pi)^{2d}} \int_{\{|\xi| < r\}} \frac{d\xi}{\sup_{x \in \mathbb{R}^d} |q(x, \xi)|} = \infty,$$

which completes the proof. \square

As a consequence, we also get the following Chung-Fuchs type conditions.

Corollary 2.3. Let $\{F_t\}_{t \geq 0}$ be a nice Feller process with symbol $q(x, \xi)$ satisfying $|\operatorname{Im} q(x, \xi)| \leq c \operatorname{Re} q(x, \xi)$ for some $c \geq 0$ and all $x, \xi \in \mathbb{R}^d$. Then,

(i) the condition in (1.3) is equivalent with

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\sup_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)} = \infty \quad \text{for some } r > 0.$$

(ii) the condition in (1.4) is equivalent with

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\inf_{x \in \mathbb{R}^d} |p(x, \xi)|} < \infty \quad \text{for some } r > 0.$$

Proof. The desired results easily follow from the following inequalities

$$\frac{\bar{c}}{\operatorname{Re} q(x, \xi)} \leq \frac{1}{\sqrt{(\operatorname{Re} q(x, \xi))^2 + (\operatorname{Im} q(x, \xi))^2}} = \frac{1}{|q(x, \xi)|} \leq \frac{1}{\operatorname{Re} q(x, \xi)},$$

where $\bar{c} = \frac{1}{\sqrt{1+c^2}}$. □

In the following proposition we discuss the dependence of the conditions in (1.3) and (1.4) on $r > 0$. First, note that if the condition in (1.4) holds for some $r_0 > 0$, then it also holds for all $0 < r \leq r_0$. In addition, if we assume that

$$\inf_{r_0 \leq |\xi| \leq r} \inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi) > 0 \quad (2.2)$$

holds for all $0 < r_0 < r$, then the condition in (1.4) does not depend on $r > 0$. In particular, (2.2) is satisfied if

$$\inf_{|\xi|=1} \inf_{x \in \mathbb{R}^d} \left(\langle \xi, c(x)\xi \rangle + \int_{\{|y| \leq \frac{1}{r}\}} \langle \xi, y \rangle^2 \nu(x, dy) \right) > 0$$

holds for all $r > r_0$. Indeed, let $r > r_0$ be arbitrary. Then, we have

$$\begin{aligned} \inf_{r_0 \leq |\xi| \leq r} \inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi) &= \inf_{r_0 \leq |\xi| \leq r} \inf_{x \in \mathbb{R}^d} \left(\frac{1}{2} \langle \xi, c(x)\xi \rangle + \int_{\mathbb{R}^d} (1 - \cos \langle \xi, y \rangle) \nu(x, dy) \right) \\ &\geq \inf_{r_0 \leq |\xi| \leq r} \inf_{x \in \mathbb{R}^d} \left(\frac{1}{2} \langle \xi, c(x)\xi \rangle + \int_{\{|\xi||y| \leq 1\}} (1 - \cos \langle \xi, y \rangle) \nu(x, dy) \right) \\ &\geq \frac{1}{\pi} \inf_{r_0 \leq |\xi| \leq r} \inf_{x \in \mathbb{R}^d} \left(\langle \xi, c(x)\xi \rangle + \int_{\{|\xi||y| \leq 1\}} \langle \xi, y \rangle^2 \nu(x, dy) \right) \\ &\geq \frac{r_0^2}{\pi} \inf_{r_0 \leq |\xi| \leq r} \inf_{x \in \mathbb{R}^d} \left(\left\langle \frac{\xi}{|\xi|}, \frac{c(x)}{|\xi|} \xi \right\rangle + \int_{\{|y| \leq \frac{1}{r}\}} \left\langle \frac{\xi}{|\xi|}, y \right\rangle^2 \nu(x, dy) \right) \\ &\geq \frac{r_0^2}{\pi} \inf_{|\xi|=1} \inf_{x \in \mathbb{R}^d} \left(\langle \xi, c(x)\xi \rangle + \int_{\{|y| \leq \frac{1}{r}\}} \langle \xi, y \rangle^2 \nu(x, dy) \right), \end{aligned}$$

where in the third step we employed the fact that $1 - \cos y \geq \frac{1}{\pi} y^2$ for all $|y| \leq \frac{\pi}{2}$.

Proposition 2.4. Let $\{F_t\}_{t \geq 0}$ be a nice Feller process with symbol $q(x, \xi)$. If the functions $\xi \mapsto \sup_{x \in \mathbb{R}^d} |q(x, \xi)|$ and $\xi \mapsto \inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)$ are radial and the function $\xi \mapsto \inf_{x \in \mathbb{R}^d} \sqrt{\operatorname{Re} q(x, \xi)}$ is subadditive, then the conditions in (1.3) and (1.4) do not depend on $r > 0$.

Proof. First, we prove that the functions $\xi \mapsto \sup_{x \in \mathbb{R}^d} |q(x, \xi)|$ and $\xi \mapsto \inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)$ are continuous. Let $\xi, \xi_0 \in \mathbb{R}^d$ be arbitrary. By [9, Lemma 3.6.21], we have

$$\begin{aligned} \left| \sqrt{\sup_{x \in \mathbb{R}^d} |q(x, \xi)|} - \sqrt{\sup_{x \in \mathbb{R}^d} |q(x, \xi_0)|} \right| &\leq \sup_{x \in \mathbb{R}^d} \left| \sqrt{|q(x, \xi)|} - \sqrt{|q(x, \xi_0)|} \right| \\ &\leq \sup_{x \in \mathbb{R}^d} \sqrt{|q(x, \xi - \xi_0)|} \end{aligned}$$

and

$$\begin{aligned} \left| \sqrt{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)} - \sqrt{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi_0)} \right| &\leq \sup_{x \in \mathbb{R}^d} \left| \sqrt{\operatorname{Re} q(x, \xi)} - \sqrt{\operatorname{Re} q(x, \xi_0)} \right| \\ &\leq \sup_{x \in \mathbb{R}^d} \sqrt{\operatorname{Re} q(x, \xi - \xi_0)} \\ &\leq \sup_{x \in \mathbb{R}^d} \sqrt{|q(x, \xi - \xi_0)|}. \end{aligned}$$

Now, by letting $\xi \rightarrow \xi_0$, the claim follows from [24, Theorem 4.4].

Let us first consider the recurrence case. Let $r_0 > 0$ be such that

$$\int_{\{|\xi| < r_0\}} \frac{d\xi}{\sup_{x \in \mathbb{R}^d} |q(x, \xi)|} < \infty$$

and

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\sup_{x \in \mathbb{R}^d} |q(x, \xi)|} = \infty$$

for all $r > r_0$. In particular, we have

$$\int_{\{r_0 \leq |\xi| \leq r\}} \frac{d\xi}{\sup_{x \in \mathbb{R}^d} |q(x, \xi)|} = \infty$$

for all $r > r_0$. Let $r > r_0$ be arbitrary. Then, by compactness, there exists a sequence $\{\xi_n\}_{n \in \mathbb{N}} \subseteq \{\xi : r_0 \leq |\xi| \leq r\}$, such that $\xi_n \rightarrow \xi_r \in \{\xi : r_0 \leq |\xi| \leq r\}$ and $\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^d} |q(x, \xi_n)| = 0$. In particular, by the continuity, $\sup_{x \in \mathbb{R}^d} |q(x, \xi_r)| = 0$. Since this is true for every $r > r_0$, we have $\lim_{r \rightarrow r_0} |\xi_r| = r_0$. Thus, by the continuity and radial property, for arbitrary $\xi_0 \in \mathbb{R}^d$, $|\xi_0| = r_0$, we have $\sup_{x \in \mathbb{R}^d} |q(x, \xi_0)| = 0$. Now, since $\xi \mapsto \sqrt{|q(x, \xi)|}$ is subadditive for all $x \in \mathbb{R}^d$ (see [9, Lemma 3.6.21]), by the radial property, for arbitrary $\xi \in \mathbb{R}^d$, we have

$$\begin{aligned} \sup_{x \in \mathbb{R}^d} \sqrt{|q(x, \xi + \xi_0)|} &\leq \sup_{x \in \mathbb{R}^d} \left(\sqrt{|q(x, \xi)|} + \sqrt{|q(x, \xi_0)|} \right) \\ &\leq \sup_{x \in \mathbb{R}^d} \sqrt{|q(x, \xi)|} + \sup_{x \in \mathbb{R}^d} \sqrt{|q(x, \xi_0)|} \\ &= \sup_{x \in \mathbb{R}^d} \sqrt{|q(x, \xi)|} \end{aligned}$$

and

$$\begin{aligned} \sup_{x \in \mathbb{R}^d} \sqrt{|q(x, \xi)|} &= \sup_{x \in \mathbb{R}^d} \sqrt{|q(x, \xi + \xi_0 - \xi_0)|} \\ &\leq \sup_{x \in \mathbb{R}^d} \left(\sqrt{|q(x, \xi + \xi_0)|} + \sqrt{|q(x, -\xi_0)|} \right) \\ &\leq \sup_{x \in \mathbb{R}^d} \sqrt{|q(x, \xi + \xi_0)|} + \sup_{x \in \mathbb{R}^d} \sqrt{|q(x, -\xi_0)|} \\ &= \sup_{x \in \mathbb{R}^d} \sqrt{|q(x, \xi + \xi_0)|}, \end{aligned}$$

that is, the function $\xi \mapsto \sup_{x \in \mathbb{R}^d} |q(x, \xi)|$ is periodic with period ξ_0 . Thus, we conclude that if (1.3) holds for some $r > 0$, then it holds for all $r > 0$.

In the transience case, by completely the same arguments as above, we have that $\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi_0) = 0$ for all $\xi_0 \in \mathbb{R}^d$, $|\xi_0| = r_0$, where $r_0 > 0$ is such that

$$\int_{\{|\xi| < r_0\}} \frac{d\xi}{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)} < \infty$$

and

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)} = \infty$$

for all $r > r_0$. Further, since we assumed that the function $\xi \mapsto \inf_{x \in \mathbb{R}^d} \sqrt{\operatorname{Re} q(x, \xi)}$ is subadditive, analogously as above, we conclude that the function $\xi \mapsto \inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)$ is periodic with period ξ_0 . Thus, if (1.4) holds for some $r > 0$, then it holds for all $r > 0$. This completes the proof of Proposition 2.4. \square

Let us remark that, in the one-dimensional case, a sufficient condition for the subadditivity of the function $\xi \mapsto \inf_{x \in \mathbb{R}} \sqrt{\operatorname{Re} q(x, \xi)}$ is the concavity of the function $|\xi| \mapsto \inf_{x \in \mathbb{R}} \sqrt{\operatorname{Re} q(x, \xi)}$. Indeed, let $\xi, \eta > 0$ be arbitrary. Then, we have

$$\begin{aligned} \inf_{x \in \mathbb{R}} \sqrt{\operatorname{Re} q(x, \xi)} &= \inf_{x \in \mathbb{R}} \sqrt{\operatorname{Re} q\left(x, \frac{\eta}{\xi + \eta} 0 + \frac{\xi}{\xi + \eta} (\xi + \eta)\right)} \\ &\geq \frac{\xi}{\xi + \eta} \inf_{x \in \mathbb{R}} \sqrt{\operatorname{Re} q(x, \xi + \eta)} \end{aligned}$$

and similarly

$$\inf_{x \in \mathbb{R}} \sqrt{\operatorname{Re} q(x, \eta)} \geq \frac{\eta}{\xi + \eta} \inf_{x \in \mathbb{R}} \sqrt{\operatorname{Re} q(x, \xi + \eta)}.$$

Thus,

$$\inf_{x \in \mathbb{R}} \sqrt{\operatorname{Re} q(x, \xi + \eta)} \leq \inf_{x \in \mathbb{R}} \sqrt{\operatorname{Re} q(x, \xi)} + \inf_{x \in \mathbb{R}} \sqrt{q(x, \eta)},$$

that is, the function $[0, \infty) \ni \xi \mapsto \inf_{x \in \mathbb{R}^d} \sqrt{\operatorname{Re} q(x, \xi)}$ is subadditive. Finally, since every non-negative and concave function is necessarily nondecreasing, the claim follows.

As we commented in the first section, in the case when a symbol $q(x, \xi)$ does not depend on the variable $x \in \mathbb{R}^d$, $\{F_t\}_{t \geq 0}$ becomes a Lévy process and, by the Lévy-Khintchine formula, we have

$$q(\xi) := q(x, \xi) = -\frac{\log \mathbb{E}^x [e^{i\langle \xi - x, F_t \rangle}]}{t} = -\frac{\log \mathbb{E}^0 [e^{i\langle \xi, F_t \rangle}]}{t}$$

and $\Phi_t(x, \xi) = e^{-tq(\xi)}$ for all $t \geq 0$ and all $x, \xi \in \mathbb{R}^d$ (see [23, Theorems 7.10 and 8.1]). Further, note that every Lévy process satisfies conditions (C1)-(C3) (see [23, Theorem 31.5]) and if $\{F_t\}_{t \geq 0}$ is a symmetric Lévy process, then $\operatorname{Re} \Phi_t(x, \xi) = \Phi_t(x, \xi) = e^{-tq(\xi)} \geq 0$. Thus, under condition (C4), we get the following well-known Chung-Fuchs conditions (see [23, Theorem 37.5]).

Corollary 2.5. *Let $\{F_t\}_{t \geq 0}$ be a Lévy process with symbol $q(\xi)$ which satisfies condition (C4). If $\{F_t\}_{t \geq 0}$ is symmetric and if*

$$\int_{\{|\xi| < r\}} \frac{d\xi}{q(\xi)} = \infty \quad \text{for some } r > 0,$$

then $\{F_t\}_{t \geq 0}$ is recurrent. If $q(\xi)$ satisfies the sector condition in (1.1) and

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\operatorname{Re} q(\xi)} < \infty \quad \text{for some } r > 0,$$

then $\{F_t\}_{t \geq 0}$ is transient.

Let us remark that in general, because of stationary and independent increments, the notion of irreducibility, and therefore condition (C4), is not needed to derive the recurrence and transience dichotomy of Lévy processes (see [23, Section 7]).

In the following theorem we give sufficient conditions for λ -irreducibility of Feller processes in terms of the symbol.

Theorem 2.6. *Let $\{F_t\}_{t \geq 0}$ be a Feller process which satisfies conditions (C1)-(C3) and such that its corresponding symbol $q(x, \xi)$ satisfies the sector condition in (1.1) and*

$$\int_{\mathbb{R}^d} \exp \left[-t \inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi) \right] d\xi < \infty \quad (2.3)$$

for all $t \geq 0$. Then, $\{F_t\}_{t \geq 0}$ possesses a density function $p(t, x, y)$, $t > 0$ and $x, y \in \mathbb{R}^d$. In addition, if $\Phi_t(x, \xi)$ is real-valued and if there exists a function $\underline{\Phi}_t(\xi)$ such that $0 < \underline{\Phi}_t(\xi) \leq \Phi_t(x, \xi)$ and $\underline{\Phi}_{s+t}(\xi) \leq \underline{\Phi}_t(\xi)$ for all $s, t \geq 0$ and all $x, \xi \in \mathbb{R}^d$, then $\{F_t\}_{t \geq 0}$ is λ -irreducible.

Proof. To prove the first claim, note that, by [28, Theorem 2.7], $\int_{\mathbb{R}^d} |\Phi_t(x, \xi)| d\xi < \infty$ for all $t > 0$ and all $x \in \mathbb{R}^d$. Thus, the following functions are well defined

$$p(t, x, y) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle \xi, y \rangle} \Phi_t(x, \xi) d\xi$$

and

$$\mathbb{P}^x(F_t \in B) = \int_{B-x} p(t, x, y) dy$$

for all $t > 0$, all $x, y \in \mathbb{R}^d$ and all $B \in \mathcal{B}(\mathbb{R}^d)$.

To prove the second claim, again by [28, Theorem 2.7], for every $t > 0$ we have

$$\begin{aligned} |p(t, x, 0) - p(t, x, y)| &= (2\pi)^{-d} \left| \int_{\mathbb{R}^d} (1 - e^{-i\langle \xi, y \rangle}) \Phi_t(x, \xi) d\xi \right| \\ &\leq (2\pi)^{-d} \int_{\mathbb{R}^d} |1 - e^{-i\langle \xi, y \rangle}| \exp \left[-\frac{t}{16} \inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi) \right] d\xi. \end{aligned}$$

Thus, by the dominated convergence theorem, for every $t_0 > 0$ the continuity of the function $y \mapsto p(t, x, y)$ at 0 is uniformly for all $t \geq t_0$ and all $x \in \mathbb{R}^d$. Further, for every $t_0 > 0$,

$$p(t, x, 0) = (2\pi)^{-d} \int_{\mathbb{R}^d} \Phi_t(x, \xi) d\xi \geq (2\pi)^{-d} \int_{\mathbb{R}^d} \underline{\Phi}_{t_0+1}(\xi) d\xi > 0$$

uniformly for all $t \in [t_0, t_0 + 1]$ and all $x \in \mathbb{R}^d$. According to this, there exists $\varepsilon := \varepsilon(t_0) > 0$ such that $p(t, x, y) > 0$ for all $t \in [t_0, t_0 + 1]$, all $x \in \mathbb{R}^d$ and all $|y| < \varepsilon$. Now, for any $n \in \mathbb{N}$, by the Chapman-Kolmogorov equation, we have that $p(t, x, y) > 0$ for all $t \in [nt_0, n(t_0 + 1)]$, all $x \in \mathbb{R}^d$ and all $|y| < n\varepsilon$. Finally, let $B \in \mathcal{B}(\mathbb{R}^d)$ such that $\lambda(B) > 0$. Then, for given $t_0 > 0$ and $x \in \mathbb{R}^d$, there exists $n := n(t_0, x) \in \mathbb{N}$, such that $\lambda((B - x) \cap \{|y| < n\varepsilon\}) > 0$, where $\varepsilon := \varepsilon(t_0) > 0$ is as above. Thus,

$$\mathbb{P}^x(F_t \in B) = \int_{B-x} p(t, x, y) dy \geq \int_{(B-x) \cap \{|y| < n\varepsilon\}} p(t, x, y) dy > 0,$$

for all $t \in [nt_0, n(t_0 + 1)]$. □

Note that the condition in (2.3) follows from the Hartman-Wintner condition in (1.2). Also, let us remark that, in the spirit of Lemma 2.2, we conjecture that a symmetric Feller process $\{F_t\}_{t \geq 0}$ with symbol $q(x, \xi)$ which satisfies conditions (C1)-(C3) also satisfies the following uniform lower bound

$$\Phi_t(x, \xi) \geq \exp \left[-ct \sup_{z \in \mathbb{R}^d} q(z, \xi) \right]$$

for all $t \geq 0$, all $x, \xi \in \mathbb{R}^d$ and some constant $c > 0$. In particular, under the condition in (2.3), this implies the λ -irreducibility of $\{F_t\}_{t \geq 0}$.

In the following corollary we derive some conditions for the recurrence and transience with respect to the dimension of the state space.

Corollary 2.7. *Let $\{F_t\}_{t \geq 0}$ be a nice Feller process with symbol $q(x, \xi)$.*

(i) *If*

$$\limsup_{|\xi| \rightarrow 0} \frac{\sup_{x \in \mathbb{R}^d} |q(x, \xi)|}{|\xi|^\alpha} < c$$

for some $\alpha > 0$ and some $c < \infty$ and if $d \leq \alpha$, then the condition in (1.3) holds true.

(ii) *If*

$$\liminf_{|\xi| \rightarrow 0} \frac{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)}{|\xi|^\alpha} > c$$

for some $\alpha > 0$ and some $c > 0$ and if $d > \alpha$, then the condition in (1.4) holds true.

Proof. (i) For $r > 0$ small enough and $d \leq \alpha$, we have

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\sup_{x \in \mathbb{R}^d} |q(x, \xi)|} \geq \frac{1}{c} \int_{\{|\xi| < r\}} \frac{d\xi}{|\xi|^\alpha} = \frac{c_d}{c} \int_0^r \rho^{d-1-\alpha} d\rho = \infty,$$

where $c_d = d\pi^{\frac{d}{2}} \Gamma\left(\frac{d}{2} + 1\right)$.

(ii) For $r > 0$ small enough and $d > \alpha$, we have

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)} \leq \frac{1}{c} \int_{\{|\xi| < r\}} \frac{d\xi}{|\xi|^\alpha} = \frac{c_d}{c} \int_0^r \rho^{d-1-\alpha} d\rho < \infty.$$

□

As a direct consequence of Corollary 2.7 we get the following conditions for the recurrence and transience with respect to the dimension of the state space.

Theorem 2.8. *Let $\{F_t\}_{t \geq 0}$ be a nice Feller process with symbol $q(x, \xi)$.*

(i) *If $\{F_t\}_{t \geq 0}$ is symmetric, $d = 1, 2$ and*

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |y|^2 \nu(x, dy) < \infty,$$

then $q(x, \xi)$ satisfies (1.3).

(ii) If $d \geq 3$ and

$$\liminf_{|\xi| \rightarrow 0} \frac{\sup_{c>0} \inf_{x \in \mathbb{R}^d} \left(\langle \xi, c(x)\xi \rangle + \int_{\{|y| \leq c\}} \langle \xi, y \rangle^2 \nu(x, dy) \right)}{|\xi|^2} > 0,$$

then $q(x, \xi)$ satisfies (1.4).

Proof. (i) The claim easily follows from the facts that $1 - \cos y \leq y^2$ for all $y \in \mathbb{R}$,

$$\frac{\sup_{x \in \mathbb{R}^d} q(x, \xi)}{|\xi|^2} \leq d \sup_{x \in \mathbb{R}^d} \max_{1 \leq i, j \leq d} |c_{ij}(x)| + \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |y|^2 \nu(x, dy)$$

for all $|\xi|$ small enough and Corollary 2.7 (i). Here we used the fact that for an arbitrary square matrix $A = (a_{ij})_{1 \leq i, j \leq d}$ and $\xi \in \mathbb{R}^d$, we have $|\langle \xi, A\xi \rangle| \leq |\xi| |A\xi| \leq d \max_{1 \leq i, j \leq d} |a_{ij}| |\xi|^2$.

(ii) The claim is an immediate consequence of the facts that $1 - \cos y \geq \frac{1}{\pi} y^2$ for all $|y| \leq \frac{\pi}{2}$ and

$$\frac{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)}{|\xi|^2} \geq \frac{\inf_{x \in \mathbb{R}^d} \left(\langle \xi, c(x)\xi \rangle + \int_{\{|y| \leq c\}} \langle \xi, y \rangle^2 \nu(x, dy) \right)}{\pi |\xi|^2}$$

for all $c > 0$ and all $|\xi|$ small enough and Corollary 2.7 (ii). \square

As in the Lévy process case, it is natural to expect that $\operatorname{Re} \Phi_t(x, \xi) = \Phi_t(x, \xi)$ if, and only if, $\{F_t\}_{t \geq 0}$ is a symmetric nice Feller process. The necessity easily follows from [28, Theorem 2.1]. On the other hand, according to [28, Theorem 2.1 and Proposition 4.6], the sufficiency holds under the assumption that $C_c^\infty(\mathbb{R}^d)$ is an operator core for the Feller generator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$, that is, $\mathcal{A}|_{C_c^\infty(\mathbb{R}^d)} = \mathcal{A}$ on $\mathcal{D}(\mathcal{A})$ (see also [3, Theorem 1]). In the recurrence case, we require that $\operatorname{Re} \Phi_t(0, \xi) \geq 0$ for all $t \geq 0$ and all $\xi \in \mathbb{R}^d$. Except in the symmetric Lévy process case, this assumption is trivially satisfied for nice Feller processes which can be obtained by the symmetrization, which has been introduced in [28]. Let $\{F_t\}_{t \geq 0}$ be a nice Feller process with symbol $q(x, \xi)$ and let $\{\bar{F}_t\}_{t \geq 0}$ be an independent copy of $\{F_t\}_{t \geq 0}$. Let us define $\tilde{F}_t := 2\bar{F}_0 - \bar{F}_t$. Then, by [28, Theorem 2.1], $\{\tilde{F}_t\}_{t \geq 0}$ is a nice Feller process with symbol $\tilde{q}(x, \xi) = q(x, \xi)$ and $\tilde{\Phi}_t(x, \xi) = \Phi_t(x, -\xi)$. Now, let us define the *symmetrization* of $\{F_t\}_{t \geq 0}$ by $F_t^s := (F_t + \tilde{F}_t)/2$. Then, by [28, Lemma 2.8], $\{F_t^s\}_{t \geq 0}$ is again a nice Feller process with symbol $q^s(x, \xi) = 2\operatorname{Re} q(x, \xi/2)$ and $\Phi_t^s = \Phi_t(x, \xi/2) \tilde{\Phi}_t(x, \xi/2) = |\Phi_t(x, \xi/2)|^2$.

Using the above symmetrization technique, we can consider the recurrence and transience property of Feller-Dynkin diffusions and stable-like processes. Recall that a *Feller-Dynkin diffusion* is a Feller process with continuous sample paths satisfying conditions (C1) and (C3). In particular, Feller-Dynkin diffusions are determined by a symbol of the form $q(x, \xi) = -i\langle \xi, b(x) \rangle + \frac{1}{2} \langle \xi, c(x)\xi \rangle$. Further, it is easy to check that the Feller generator $(\mathcal{A}, \mathcal{D}_\mathcal{A})$ of Feller-Dynkin diffusions restricted to $C_c^\infty(\mathbb{R}^d)$ is a second-order elliptic operator

$$\mathcal{A}|_{C_c^\infty(\mathbb{R}^d)} f(x) = \sum_{i=1}^d b_i(x) \frac{\partial f(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d c_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j}.$$

Hence, since Feller-Dynkin diffusions are Feller process, $\mathcal{A}(C_c^\infty(\mathbb{R}^d)) \subseteq C_\infty(\mathbb{R}^d)$, therefore $b(x)$ and $c(x)$ are continuous functions. For further properties of Feller-Dynkin diffusions see [16] and [17].

Theorem 2.9. *Let $\{F_t\}_{t \geq 0}$ be a Feller-Dynkin diffusion with symbol $q(x, \xi) = -i\langle \xi, b(x) \rangle + \frac{1}{2} \langle \xi, c(x)\xi \rangle$ which satisfies condition (C2). Further, assume that $\inf_{x \in \mathbb{R}^d} \langle \xi, c(x)\xi \rangle \geq c|\xi|^2$ for all $\xi \in \mathbb{R}^d$ and some $c > 0$ and that $b(x)$ and $c(x)$ are Hölder continuous with the index $0 < \beta \leq 1$.*

- (i) If $\{F_t\}_{t \geq 0}$ is symmetric and $d = 1, 2$, then $\{F_t\}_{t \geq 0}$ is recurrent.
- (ii) If $d \geq 3$, then $\{F_t\}_{t \geq 0}$ is transient.

Proof. First, let us remark that, by [29, Theorem A], $\{F_t\}_{t \geq 0}$ possesses a strictly positive density function. In particular, $\{F_t\}_{t \geq 0}$ is λ -irreducible, that is, it satisfies condition (C4). Hence, $\{F_t\}_{t \geq 0}$ is a nice Feller process.

- (i) Let $\{\bar{F}_t\}_{t \geq 0}$ be a Feller-Dynkin diffusion with symbol given by $\bar{q}(x, \xi) = \langle \xi, c(x)\xi \rangle$. The existence (and uniqueness) of the process $\{\bar{F}_t\}_{t \geq 0}$ is given in [17, Theorem 24.1]. Again, $\{\bar{F}_t\}_{t \geq 0}$ is a symmetric nice Feller process. Now, by the symmetrization and [17, Theorem 24.1], $\{\bar{F}_t^s\}_{t \geq 0} \stackrel{d}{=} \{F_t\}_{t \geq 0}$ and, in particular, $\Phi_t(x, \xi) \geq 0$ for all $t \geq 0$ and all $x, \xi \in \mathbb{R}^d$. Now, the claim easily follows from Theorem 2.8 (i).
- (ii) This is an immediate consequence of Theorems 1.3 and 2.8 (ii).

□

Let $\alpha : \mathbb{R}^d \rightarrow (0, 2)$ and $\gamma : \mathbb{R}^d \rightarrow (0, \infty)$ be arbitrary bounded and continuously differentiable functions with bounded derivatives, such that $0 < \underline{\alpha} := \inf_{x \in \mathbb{R}^d} \alpha(x) \leq \sup_{x \in \mathbb{R}^d} \alpha(x) =: \bar{\alpha} < 2$ and $\inf_{x \in \mathbb{R}^d} \gamma(x) > 0$. Under this assumptions, in [1] and [28, Theorem 3.3.] it has been shown that there exists a unique nice Feller process, called a *stable-like process*, determined by a symbol of the form $p(x, \xi) = \gamma(x)|\xi|^{\alpha(x)}$. The λ -irreducibility follows from Theorem 2.6 and [11, Theorem 5.1]. Further, note that when $\alpha(x)$ and $\gamma(x)$ are constant functions, then we deal with a rotationally invariant stable Lévy process. Now, as a direct consequence of Theorems 1.2 and 1.3 and Corollary 2.7, we get the following conditions for recurrence and transience of stable-like processes.

Theorem 2.10. *Let $\{F_t\}_{t \geq 0}$ be a stable-like process with symbol given by $q(x, \xi) = \gamma(x)|\xi|^{\alpha(x)}$.*

- (i) *If $\{F_t\}_{t \geq 0}$ is one-dimensional and if $\inf_{x \in \mathbb{R}} \alpha(x) \geq 1$, then $\{F_t\}_{t \geq 0}$ is recurrent.*
- (ii) *If $\{F_t\}_{t \geq 0}$ is one-dimensional and if $\sup_{x \in \mathbb{R}} \alpha(x) < 1$, then $\{F_t\}_{t \geq 0}$ is transient.*
- (iii) *If $d \geq 2$, then $\{F_t\}_{t \geq 0}$ is transient.*

Proof. Let $\{\bar{F}_t\}_{t \geq 0}$ be a stable-like process determined by a symbol of the form

$$\bar{q}(x, \xi) = 2^{\alpha(x)-1} \gamma(x) |\xi|^{\alpha(x)}.$$

Then, by the symmetrization, $\{\bar{F}_t^s\}_{t \geq 0} \stackrel{d}{=} \{F_t\}_{t \geq 0}$ and, in particular, $\Phi_t(x, \xi) \geq 0$ for all $t \geq 0$ and all $x, \xi \in \mathbb{R}^d$. Now, the desired results easily follow from Theorems 1.2 and 1.3 and Corollary 2.7. □

Let us remark that the recurrence and transience property of stable-like processes has been studied extensively in the literature (see [2], [6], [18], [19], [20], [21], [28]).

In what follows, we briefly discuss the recurrence and transience property of symmetric nice Feller processes obtained by variable order subordination (see also [28]). Let $q : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous and negative definite function such that $q(0) = 0$ (that is, $q(\xi)$ is the symbol of some symmetric Lévy process). Further, let $f : \mathbb{R}^d \times [0, \infty) \rightarrow [0, \infty)$ be a measurable function such that $\sup_{x \in \mathbb{R}^d} f(x, t) \leq c(1 + t)$ for some $c \geq 0$ and all $t \in [0, \infty)$, and for fixed $x \in \mathbb{R}^d$ the function $t \rightarrow f(x, t)$ is a Bernstein function with $f(x, 0) = 0$. Bernstein functions are the characteristic Laplace exponents of subordinators (Lévy processes with nondecreasing sample paths). For more

on Bernstein functions we refer the readers to the monograph [26]. Now, since $q(\xi) \geq 0$ for all $\xi \in \mathbb{R}^d$, the function

$$\bar{q}(x, \xi) := f(x, q(\xi))$$

is well defined and, according to [26, Theorem 5.2] and [23, Theorem 30.1], $\xi \mapsto \bar{q}(x, \xi)$ is a continuous and negative definite function satisfying conditions **(C2)** and **(C3)**. Hence, $\bar{q}(x, \xi)$ is possibly the symbol of some symmetric Feller process. Of special interest is the case when $f(x, t) = t^{\alpha(x)}$, where $\alpha : \mathbb{R}^d \mapsto [0, 1]$, that is, $\bar{q}(x, \xi)$ describes variable order subordination. For sufficient conditions on the symbol $q(\xi)$ and function $\alpha(x)$ such that $\bar{q}(x, \xi)$ is the symbol of some Feller process see [5] and [8] and the references therein. Now, let $0 \leq \underline{\alpha} := \inf_{x \in \mathbb{R}^d} \alpha(x) \leq \sup_{x \in \mathbb{R}^d} \alpha(x) =: \bar{\alpha} \leq 1$. Then, since the symbol $q(\xi)$ is continuous and $q(0) = 0$, there exists $r > 0$ small enough, such that $q(\xi) \leq 1$ for all $|\xi| < r$. In particular, the conditions in (1.3) and (1.4) hold true if

$$\int_{\{|\xi| < r\}} \frac{d\xi}{(q(\xi))^{\underline{\alpha}}} = \infty \quad \text{and} \quad \int_{\{|\xi| < r\}} \frac{d\xi}{(q(\xi))^{\bar{\alpha}}} < \infty,$$

respectively (see also [28, Corollary 3.2]). Note that when $q(\xi)$ is the symbol of a Brownian motion, then by variable order subordination we get a stable-like process.

If we know the distribution of $\{F_t\}_{t \geq 0}$, in order to prove the recurrence of $\{F_t\}_{t \geq 0}$, the assumption $\operatorname{Re} \Phi_t(0, \xi) \geq 0$ for all $t \geq 0$ and all $\xi \in \mathbb{R}^d$ can be replaced by the following assumptions.

Proposition 2.11. *Let $\{F_t\}_{t \geq 0}$ be a nice Feller process with symbol $q(x, \xi)$ which satisfies the condition in (1.3). If there exists $x_0 \in \mathbb{R}^d$ such that for every $a > 0$ there exist $b > 0$, $\varepsilon > 0$ and $t_0 \geq 0$, such that*

$$\mathbb{P}^{x_0}(F_t \in B_a(x_0)) \geq \varepsilon \sup_{y \in \mathbb{R}^d} \mathbb{P}^{x_0}(F_t \in B_b(2x_0 - y))$$

for all $t \geq t_0$, then $\{F_t\}_{t \geq 0}$ is recurrent. Here, $B_r(x) := \{z : |z - x| < r\}$ denotes the open ball of radius $r > 0$ around $x \in \mathbb{R}^d$. In addition, if $q(x, \xi)$ satisfies the condition in (2.3), then $\{F_t\}_{t \geq 0}$ is recurrent if there exists $x_0 \in \mathbb{R}^d$ such that for every $a > 0$ there exist $\varepsilon > 0$ and $t_0 \geq 0$, such that

$$\mathbb{P}^{x_0}(F_t \in B_a(x_0)) \geq \varepsilon \int_{\mathbb{R}^d} \exp \left[-t \inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi) \right] d\xi$$

for all $t \geq t_0$.

Proof. Let $\{F_t^s\}_{t \geq 0}$ be the symmetrization of $\{F_t\}_{t \geq 0}$. Then, by assumption, $\{F_t^s\}_{t \geq 0}$ is recurrent. Next, in order to prove the recurrence of $\{F_t\}_{t \geq 0}$, by Proposition 2.1 (iv), it suffices to show that there exists $x \in \mathbb{R}^d$ such that for every $a > 0$ we have

$$\int_0^\infty \mathbb{P}^x(F_t \in B_a(x)) = \infty.$$

Let $a > 0$ be arbitrary and let $x_0 \in \mathbb{R}^d$, $b > 0$, $\varepsilon > 0$ and $t_0 \geq 0$ be as above. Then, for $t \geq t_0$ we have

$$\mathbb{P}^{x_0}(F_t^s \in B_{b/2}(x_0)) = \int_{\mathbb{R}^d} \mathbb{P}^{x_0}(F_t \in B_b(2x_0 - y)) \mathbb{P}^{x_0}(F_t \in dy) \leq \frac{\mathbb{P}^{x_0}(F_t \in B_a(x_0))}{\varepsilon}.$$

To prove the second part, note that, by Theorem 2.6, $\mathbb{P}^x(F_t \in dy) = p(t, x, y)dy$, for $t > 0$ and $x, y \in \mathbb{R}^d$. Thus, by [28, Theorem 1.1], we have

$$\mathbb{P}^{x_0}(F_t \in B_b(2x_0 - y)) = \int_{B_b(2x_0 - y) - x_0} p(t, x_0, z) dz \leq \frac{\lambda(B_b(0))}{(4\pi)^d} \int_{\mathbb{R}^d} \exp \left[-\frac{t}{16} \inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi) \right] d\xi$$

for all $t > 0$ and all $y \in \mathbb{R}^d$, where $x_0 \in \mathbb{R}^d$ and $b > 0$ are as above. \square

In Section 1, we recalled the notion of stable Lévy processes and the indices of stability. The concept of the indices of stability can be generalized to general Lévy process through the so-called Pruitt indices (see [15]). The *Pruitt indices*, for a nice Feller process $\{F_t\}_{t \geq 0}$ with symbol $q(x, \xi)$, are defined in the following way

$$\begin{aligned}\beta &:= \sup \left\{ \alpha \geq 0 : \lim_{|\xi| \rightarrow 0} \frac{\sup_{x \in \mathbb{R}^d} |q(x, \xi)|}{|\xi|^\alpha} = 0 \right\} \\ \delta &:= \sup \left\{ \alpha \geq 0 : \lim_{|\xi| \rightarrow 0} \frac{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)}{|\xi|^\alpha} = 0 \right\}\end{aligned}$$

(see [25]). Note that $0 \leq \beta \leq \delta$, $\beta \leq 2$ and in the case of a stable-like process with symbol given by $q(x, \xi) = \gamma(x)|\xi|^{\alpha(x)}$, we have $\beta = \underline{\alpha}$ and $\delta = \bar{\alpha}$. Now, we generalize Theorem 2.10 in terms of the Pruitt indices.

Theorem 2.12. *Let $\{F_t\}_{t \geq 0}$ be a nice Feller process with symbol $q(x, \xi)$.*

(i) *If $d = 1$ and $\beta > 1$, then the condition in (1.3) holds true.*

(ii) *If $q(x, \xi)$ satisfies the condition in (1.4), then $\delta \leq d$.*

Proof. (i) Let $1 \leq \alpha < \beta$ be arbitrary. Then, by the definition of β ,

$$\lim_{|\xi| \rightarrow 0} \frac{\sup_{x \in \mathbb{R}} |q(x, \xi)|}{|\xi|^\alpha} = 0.$$

Now, the claim easily follows from Corollary 2.7 (i).

(ii) Let us assume that this is not the case. Then, for all $d \leq \alpha < \delta$, by the definition of δ , we have

$$\lim_{|\xi| \rightarrow 0} \frac{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)}{|\xi|^\alpha} = 0.$$

By Corollary 2.7 (i), this yields that

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)} \geq \int_{\{|\xi| < r\}} \frac{d\xi}{|\xi|^\alpha} = \infty$$

for all $r > 0$ small enough. □

Let us remark that, according to Theorem 2.10, in the dimension $d \geq 2$ even under the condition in (1.4) we can have $\delta \geq 1$. Similarly, the recurrence of a one-dimensional symmetric nice Feller process, in general, does not imply that $\beta \geq 1$. To see this, first recall that for two symmetric measures $\mu(dy)$ and $\bar{\mu}(dy)$ on $\mathcal{B}(\mathbb{R})$ which are finite outside of any neighborhood of the origin, we say that $\mu(dy)$ has a bigger tail than $\bar{\mu}(dy)$ if there exists $y_0 > 0$ such that $\mu(y, \infty) \geq \bar{\mu}(y, \infty)$ for all $y \geq y_0$. Now, by [23, Theorem 38.4], if $\bar{\nu}(dy)$ is the Lévy measure of a transient one-dimensional symmetric Lévy process $\{\bar{L}_t\}_{t \geq 0}$, then there exists a recurrent one-dimensional symmetric Lévy process $\{L_t\}_{t \geq 0}$ with Lévy measure $\nu(dy)$ having a bigger tail than $\bar{\nu}(dy)$. Further, by Fubini's theorem, for any $\alpha > 0$ we have

$$\begin{aligned}\int_{\{y > y_0\}} y^\alpha \nu(dy) &= \int_{\{y > y_0\}} \int_0^y \alpha z^{\alpha-1} dz \nu(dy) \\ &= y_0^\alpha \nu(y_0, \infty) + \alpha \int_{\{z > y_0\}} z^{\alpha-1} \nu(z, \infty) dz \\ &\geq y_0^\alpha \nu(y_0, \infty) + \alpha \int_{\{z > y_0\}} z^{\alpha-1} \bar{\nu}(z, \infty) dz.\end{aligned}$$

Hence, if $\int_{\{y>1\}} y^\alpha \bar{\nu}(dy) = \infty$, then $\int_{\{y>1\}} y^\alpha \nu(dy) = \infty$. Therefore, by [25, Proposition 5.4], the recurrence of $\{L_t\}_{t \geq 0}$ does not imply that $\beta \geq 1$. Similarly, by [23, Theorem 38.4] and [25, Proposition 5.4], in the one-dimensional symmetric case, $\delta \leq 1$ does not automatically imply transience. By assuming certain regularities (convexity and concavity) on symbol $q(x, \xi)$, we get the converse of Theorem 2.12 (see [23, Theorem 38.2] for the Lévy process case).

Theorem 2.13. *Let $\{F_t\}_{t \geq 0}$ be a nice Feller process with symbol $q(x, \xi)$.*

- (i) *If $d = 1$, the function $\xi \mapsto \sup_{x \in \mathbb{R}} |q(x, \xi)|$ is radial and convex on some neighborhood around the origin and $q(x, \xi)$ satisfies the condition in (1.3), then $\beta \geq 1$.*
- (ii) *If the function $\xi \mapsto \sup_{x \in \mathbb{R}^d} |q(x, \xi)|$ is radial and concave on some neighborhood around the origin, then $\beta \leq 1$.*
- (iii) *If the function $\xi \mapsto \inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)$ is radial and concave on some neighborhood around the origin, $d \geq 2$ and $\delta < d$, then $q(x, \xi)$ satisfies the condition in (1.4).*
- (iv) *If the function $\xi \mapsto \inf_{x \in \mathbb{R}^d} \operatorname{Re} p(x, \xi)$ is radial and convex on some neighborhood around the origin, then $\delta \geq 1$.*

Proof. (i) We show that

$$\lim_{|\xi| \rightarrow 0} \frac{\sup_{x \in \mathbb{R}} |q(x, \xi)|}{|\xi|^\alpha} = 0$$

for all $\alpha < 1$. Let us assume that this is not the case. Then, there exists $\alpha_0 < 1$ such that

$$\limsup_{|\xi| \rightarrow 0} \frac{\sup_{x \in \mathbb{R}} |q(x, \xi)|}{|\xi|^{\alpha_0}} > c$$

for some $c > 0$. Hence, there exists a sequence $\{\xi_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ such that $\lim_{n \rightarrow \infty} |\xi_n| = 0$ and

$$\lim_{n \rightarrow \infty} \frac{\sup_{x \in \mathbb{R}} |q(x, \xi_n)|}{|\xi_n|^{\alpha_0}} > c.$$

Thus, $\sup_{x \in \mathbb{R}} |q(x, \xi_n)| \geq c|\xi_n|^{\alpha_0}$ for all $n \in \mathbb{N}$ large enough. Now, because of the radial symmetry and convexity assumptions, $\sup_{x \in \mathbb{R}} |q(x, \xi)| \geq c|\xi|^{\alpha_0}$ for all $|\xi|$ small enough. Indeed, if this was not the case, then there would exist a sequence $\{\bar{\xi}_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ such that $\lim_{n \rightarrow \infty} |\bar{\xi}_n| = 0$ and $\sup_{x \in \mathbb{R}^d} |q(x, \bar{\xi}_n)| < c|\bar{\xi}_n|^{\alpha_0}$ for all $n \in \mathbb{N}$. Now, let $n, m \in \mathbb{N}$ be such that $|\xi_n| < |\bar{\xi}_m|$ and $\sup_{x \in \mathbb{R}} |q(x, \xi_n)| \geq c|\xi_n|^{\alpha_0}$. Then, for adequately chosen $t \in [0, 1]$, we have

$$c|\xi_n|^{\alpha_0} \leq \sup_{x \in \mathbb{R}} |q(x, \xi_n)| = \sup_{x \in \mathbb{R}} |q(x, t\bar{\xi}_m)| \leq t \sup_{x \in \mathbb{R}} |q(x, \bar{\xi}_m)| < ct|\bar{\xi}_m|^{\alpha_0} \leq ct^{\alpha_0} |\bar{\xi}_m|^{\alpha_0} = c|\xi_n|^{\alpha_0},$$

where in the third step we took into account the convexity property. Hence, $\sup_{x \in \mathbb{R}} |q(x, \xi)| \geq c|\xi|^{\alpha_0}$ for all $|\xi|$ small enough. But, according to Proposition 2.4, this is in contradiction with the condition in (1.3).

- (ii) Let $\varepsilon > 0$ be such that $|\xi| \mapsto \sup_{x \in \mathbb{R}^d} |q(x, \xi)|$ is concave on $[0, \varepsilon)$. Now, for all $\alpha \geq 1$, we have

$$\begin{aligned} \liminf_{|\xi| \rightarrow 0} \frac{\sup_{x \in \mathbb{R}^d} |q(x, \xi)|}{|\xi|^\alpha} &= \liminf_{|\xi| \rightarrow 0} \frac{\sup_{x \in \mathbb{R}^d} \left| q\left(x, \frac{2|\xi|}{\varepsilon} \frac{\varepsilon \xi}{2|\xi|}\right) \right|}{|\xi|^\alpha} \\ &\geq \liminf_{|\xi| \rightarrow 0} \frac{2|\xi|^{1-\alpha}}{\varepsilon} \sup_{x \in \mathbb{R}^d} \left| q\left(x, \frac{\varepsilon \xi}{2|\xi|}\right) \right| > 0, \end{aligned}$$

where in the second step we applied the concavity property. Now, the desired result follows from the definition of the index β .

(iii) Let $\max\{1, \delta\} < \alpha < d$ be arbitrary. By the definition of δ , we have

$$\limsup_{|\xi| \rightarrow 0} \frac{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)}{|\xi|^\alpha} > c$$

for some $c > 0$. Thus, there exists a sequence $\{\xi_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^d$ such that $\lim_{n \rightarrow \infty} |\xi_n| = 0$ and

$$\lim_{n \rightarrow \infty} \frac{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi_n)}{|\xi_n|^\alpha} > c.$$

In particular, $\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi_n) \geq c|\xi_n|^\alpha$ for all $n \in \mathbb{N}$ large enough. Actually, because of the radial symmetry and concavity assumptions, $\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi) \geq c|\xi|^\alpha$ for all $|\xi|$ small enough. Indeed, for all $n \in \mathbb{N}$ large enough and all $t \in [0, 1]$ we have

$$\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, t\xi_n) \geq t \inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi_n) \geq ct|\xi_n|^\alpha \geq ct^\alpha |\xi_n|^\alpha = c|t\xi_n|^\alpha,$$

where in the first step we took into account the concavity property. Now, the claim is a direct consequence of Corollary 2.7 (ii).

(iv) Let $\varepsilon > 0$ be such that $|\xi| \mapsto \inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)$ is convex on $[0, \varepsilon]$ and let $\alpha < 1$ be arbitrary. Then, we have

$$\begin{aligned} \limsup_{|\xi| \rightarrow 0} \frac{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)}{|\xi|^\alpha} &= \limsup_{|\xi| \rightarrow 0} \frac{\inf_{x \in \mathbb{R}^d} \operatorname{Re} q\left(x, \frac{2|\xi|}{\varepsilon} \frac{\varepsilon \xi}{2|\xi|}\right)}{|\xi|^\alpha} \\ &\leq \limsup_{|\xi| \rightarrow 0} \frac{2 \inf_{x \in \mathbb{R}^d} \operatorname{Re} q\left(x, \frac{\varepsilon \xi}{2|\xi|}\right)}{\varepsilon |\xi|^{\alpha-1}} = 0, \end{aligned}$$

where in the second step we employed the convexity property. Now, the desired result follows from the definition of the index δ . □

Also, let us remark that the conclusions of Theorem 2.13 can be easily obtained if instead of the convexity and concavity and radial symmetry assumptions on the functions $\xi \mapsto \sup_{x \in \mathbb{R}^d} |q(x, \xi)|$ and $\xi \mapsto \inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi)$ we assume that

$$c^{-1}|\xi|^\alpha \leq \sup_{x \in \mathbb{R}^d} |q(x, \xi)| \leq c|\xi|^\beta \quad \text{and} \quad c^{-1}|\xi|^\alpha \leq \inf_{x \in \mathbb{R}^d} \operatorname{Re} q(x, \xi) \leq c|\xi|^\beta$$

for all $|\xi|$ small enough and some adequate $0 < \beta \leq \alpha < \infty$ and $c > 0$.

3 Recurrence and transience of one-dimensional symmetric nice Feller processes

In this section, we consider the recurrence and transience property of one-dimensional symmetric nice Feller processes. Note that in this case Proposition 2.4 holds true, that is, the condition in (1.3) does not depend on $r > 0$. On the other hand, recall that if the condition in (1.4) holds for

some $r_0 > 0$, then it also holds for all $r > r_0$. In situations where we need complete independence of $r > 0$, we assume the subadditivity of the function $\xi \mapsto \inf_{x \in \mathbb{R}} \sqrt{q(x, \xi)}$ (see Proposition 2.4).

First, we study perturbations of symbols which do not affect the recurrence and transience property of the underlying Feller process.

Theorem 3.1. *Let $\{F_t^1\}_{t \geq 0}$ and $\{F_t^2\}_{t \geq 0}$ be one-dimensional symmetric nice Feller processes with symbols $q_1(x, \xi)$ and $q_2(x, \xi)$ and Lévy measures $\nu_1(x, dy)$ and $\nu_2(x, dy)$, respectively. If $q_1(x, \xi)$ satisfies (1.3) and*

$$\sup_{x \in \mathbb{R}} \int_0^\infty y^2 |\nu_1(x, dy) - \nu_2(x, dy)| < \infty, \quad (3.1)$$

then $q_2(x, \xi)$ also satisfies (1.3). Further, if $q_1(x, \xi)$ satisfies (1.4),

$$\lim_{\xi \rightarrow 0} \frac{\inf_{x \in \mathbb{R}} q_1(x, \xi)}{\xi^2} = \infty \quad (3.2)$$

and (3.1), then $q_2(x, \xi)$ also satisfies (1.4) and (3.2). Here, $|\mu(dy)|$ denotes the total variation measure of the signed measure $\mu(dy)$.

Proof. Let

$$q_1(x, \xi) = \frac{1}{2} c_1(x) \xi^2 + \int_{\mathbb{R}} (1 - \cos \xi y) \nu_1(x, dy) \quad \text{and} \quad q_2(x, \xi) = \frac{1}{2} c_2(x) \xi^2 + \int_{\mathbb{R}} (1 - \cos \xi y) \nu_2(x, dy)$$

be the symbols of $\{F_t^1\}_{t \geq 0}$ and $\{F_t^2\}_{t \geq 0}$, respectively. First, let us prove the recurrence case. Note that (3.1) implies that

$$\sup_{x \in \mathbb{R}} \int_0^\infty y^2 \nu_1(x, dy) < \infty \quad \text{if, and only if,} \quad \sup_{x \in \mathbb{R}} \int_0^\infty y^2 \nu_2(x, dy) < \infty.$$

Indeed, we have

$$\begin{aligned} \int_0^\infty y^2 \nu_1(x, dy) &= \int_0^\infty y^2 |\nu_1(x, dy) - \nu_2(x, dy) + \nu_2(x, dy)| \\ &\leq \int_0^\infty y^2 \nu_2(x, dy) + \int_0^\infty y^2 |\nu_1(x, dy) - \nu_2(x, dy)| \end{aligned}$$

and similarly

$$\int_0^\infty y^2 \nu_2(x, dy) \leq \int_0^\infty y^2 \nu_1(x, dy) + \int_0^\infty y^2 |\nu_1(x, dy) - \nu_2(x, dy)|.$$

Now, in the case when $\sup_{x \in \mathbb{R}} \int_0^\infty y^2 \nu_1(x, dy) < \infty$, the claim easily follows from Theorem 2.8 (i). Suppose that $\sup_{x \in \mathbb{R}} \int_0^\infty y^2 \nu_1(x, dy) = \infty$. Then, by Fatou's lemma, we have

$$\liminf_{\xi \rightarrow 0} \sup_{x \in \mathbb{R}} \int_0^\infty \frac{1 - \cos \xi y}{\xi^2} \nu_1(x, dy) \geq \liminf_{\xi \rightarrow 0} \int_0^\infty \frac{1 - \cos \xi y}{\xi^2} \nu_1(x, dy) = \frac{1}{2} \int_0^\infty y^2 \nu_1(x, dy).$$

Hence,

$$\lim_{\xi \rightarrow 0} \frac{\sup_{x \in \mathbb{R}} q_1(x, \xi)}{\xi^2} = \infty. \quad (3.3)$$

Next, we have

$$\begin{aligned}
& \left| \sup_{x \in \mathbb{R}} q_1(x, \xi) - \sup_{x \in \mathbb{R}} q_2(x, \xi) \right| \\
& \leq \sup_{x \in \mathbb{R}} |q_1(x, \xi) - q_2(x, \xi)| \\
& \leq \frac{1}{2} \sup_{x \in \mathbb{R}} |c_1(x) - c_2(x)| \xi^2 + 2 \sup_{x \in \mathbb{R}} \left| \int_0^\infty (1 - \cos \xi y) \nu_1(x, dy) - \int_0^\infty (1 - \cos \xi y) \nu_2(x, dy) \right| \\
& \leq \frac{1}{2} \sup_{x \in \mathbb{R}} |c_1(x) - c_2(x)| \xi^2 + 2 \sup_{x \in \mathbb{R}} \int_0^\infty (1 - \cos \xi y) |\nu_1(x, dy) - \nu_2(x, dy)| \\
& \leq \frac{1}{2} \sup_{x \in \mathbb{R}} |c_1(x) - c_2(x)| \xi^2 + 2 \xi^2 \sup_{x \in \mathbb{R}} \int_0^\infty y^2 |\nu_1(x, dy) - \nu_2(x, dy)| \\
& \leq \left(\frac{1}{2} \sup_{x \in \mathbb{R}} |c_1(x) - c_2(x)| + 2 \sup_{x \in \mathbb{R}} \int_0^\infty y^2 |\nu_1(x, dy) - \nu_2(x, dy)| \right) \xi^2, \tag{3.4}
\end{aligned}$$

where in the fourth step we used the fact that $1 - \cos y \leq y^2$ for all $y \in \mathbb{R}$. Finally, by (3.3) and (3.4), we have

$$\lim_{\xi \rightarrow 0} \frac{\sup_{x \in \mathbb{R}} q_2(x, \xi)}{\sup_{x \in \mathbb{R}} q_2(x, \xi)} = 1 + \lim_{\xi \rightarrow 0} \frac{\sup_{x \in \mathbb{R}} q_2(x, \xi) - \sup_{x \in \mathbb{R}} q_1(x, \xi)}{\sup_{x \in \mathbb{R}} q_1(x, \xi)} = 1,$$

which together with Proposition 2.4 proves the claim.

In the transience case, we proceed in the similar way. We have

$$\begin{aligned}
& \left| \inf_{x \in \mathbb{R}} q_1(x, \xi) - \inf_{x \in \mathbb{R}} q_2(x, \xi) \right| \\
& \leq \sup_{x \in \mathbb{R}} |q_1(x, \xi) - q_2(x, \xi)| \\
& \leq \left(\frac{1}{2} \sup_{x \in \mathbb{R}} |c_1(x) - c_2(x)| + 2 \sup_{x \in \mathbb{R}} \int_0^\infty y^2 |\nu_1(x, dy) - \nu_2(x, dy)| \right) \xi^2. \tag{3.5}
\end{aligned}$$

Hence, by (3.2) and (3.5), we have

$$\lim_{\xi \rightarrow 0} \frac{\inf_{x \in \mathbb{R}} q_2(x, \xi)}{\inf_{x \in \mathbb{R}} q_1(x, \xi)} = 1 + \lim_{\xi \rightarrow 0} \frac{\inf_{x \in \mathbb{R}} q_2(x, \xi) - \inf_{x \in \mathbb{R}} q_1(x, \xi)}{\inf_{x \in \mathbb{R}} q_1(x, \xi)} = 1$$

and

$$\lim_{\xi \rightarrow 0} \frac{\inf_{x \in \mathbb{R}} q_2(x, \xi)}{\xi^2} = \infty.$$

Now, by applying Proposition 2.4, the claim follows. \square

Let us remark that it is easy to see that the condition in (3.2) can be relaxed to the following condition

$$\liminf_{\xi \rightarrow 0} \frac{\inf_{x \in \mathbb{R}} q_1(x, \xi)}{\xi^2} > \frac{1}{2} \sup_{x \in \mathbb{R}} |c_1(x) - c_2(x)| + 2 \sup_{x \in \mathbb{R}} \int_0^\infty y^2 |\nu_1(x, dy) - \nu_2(x, dy)|.$$

Theorem 3.1 essentially says that, in the one-dimensional symmetric case, the recurrence and transience property of nice Feller processes depends only on big jumps. A situation where the perturbation condition in (3.1) easily holds true is given in the following proposition.

Proposition 3.2. Let $\{F_t^1\}_{t \geq 0}$ and $\{F_t^2\}_{t \geq 0}$ be one-dimensional symmetric nice Feller processes with Lévy measures $\nu_1(x, dy)$ and $\nu_2(x, dy)$, respectively. If there exists $y_0 > 0$ such that $\nu_1(x, (y, \infty)) = \nu_2(x, (y, \infty))$ for all $x \in \mathbb{R}$ and all $y \geq y_0$, then the condition in (3.1) holds true.

Now, as a simple consequence of Theorem 3.1 and Proposition 3.2 we can generalize Theorem 2.10 (see also [2, Theorem 4.6]).

Corollary 3.3. Let $\{F_t\}_{t \geq 0}$ be a one-dimensional stable-like process with symbol $q(x, \xi) = \gamma(x)|\xi|^{\alpha(x)}$.

(i) If $\liminf_{|x| \rightarrow \infty} \alpha(x) \geq 1$, then $\{F_t\}_{t \geq 0}$ is recurrent.

(ii) If $\limsup_{|x| \rightarrow \infty} \alpha(x) < 1$, then $\{F_t\}_{t \geq 0}$ is transient.

Let us remark that by allowing $\liminf_{|x| \rightarrow \infty} \alpha(x) = 1$, the above corollary also generalizes [21, Theorem 1.3], [19, Theorem 1.3] and [20, Theorem 1.1]. In the following theorem, we slightly generalize Proposition 3.2.

Theorem 3.4. Let $\{F_t^1\}_{t \geq 0}$ and $\{F_t^2\}_{t \geq 0}$ be one-dimensional symmetric nice Feller processes with symbols $q_1(x, \xi)$ and $q_2(x, \xi)$ and Lévy measures $\nu_1(x, dy)$ and $\nu_2(x, dy)$, respectively. Further, assume that there exists a compact set $C \subseteq \mathbb{R}$ such that $\nu_1(x, B \cap C^c) \geq \nu_2(x, B \cap C^c)$ for all $x \in \mathbb{R}$ and all $B \in \mathcal{B}(\mathbb{R})$. If $q_1(x, \xi)$ satisfies (1.3), then $q_2(x, \xi)$ also satisfies (1.3). Next, if $q_2(x, \xi)$ satisfies (1.4) and (3.2), then $q_1(x, \xi)$ also satisfies (1.4) and (3.2).

Proof. Let

$$q_1(x, \xi) = \frac{1}{2}c_1(x)\xi^2 + \int_{\mathbb{R}} (1 - \cos \xi y) \nu_1(x, dy) \quad \text{and} \quad q_2(x, \xi) = \frac{1}{2}c_2(x)\xi^2 + \int_{\mathbb{R}} (1 - \cos \xi y) \nu_2(x, dy)$$

be the symbols of $\{F_t^1\}_{t \geq 0}$ and $\{F_t^2\}_{t \geq 0}$, respectively. First, let us prove the recurrence case. Let $m > 0$ be so large that $C \subseteq [-m, m]$. We have

$$\begin{aligned} q_2(x, \xi) &= \frac{1}{2}c_2(x)\xi^2 + \int_{\mathbb{R}} (1 - \cos \xi y) \nu_2(x, dy) \\ &= \frac{1}{2}c_2(x)\xi^2 + \int_{[-m, m]} (1 - \cos \xi y) \nu_2(x, dy) + \int_{[-m, m]^c} (1 - \cos \xi y) \nu_2(x, dy) \\ &\leq \frac{1}{2}c_2(x)\xi^2 + \int_{[-m, m]} (1 - \cos \xi y) \nu_2(x, dy) + \int_{[-m, m]^c} (1 - \cos \xi y) \nu_1(x, dy) \\ &\leq \frac{1}{2} \sup_{x \in \mathbb{R}} c_2(x) \xi^2 + \xi^2 \sup_{x \in \mathbb{R}} \int_{[-m, m]} y^2 \nu_2(x, dy) + \int_{[-m, m]^c} (1 - \cos \xi y) \nu_1(x, dy) \\ &\leq \left(\frac{1}{2} \sup_{x \in \mathbb{R}} c_2(x) + \sup_{x \in \mathbb{R}} \int_{[-m, m]} y^2 \nu_2(x, dy) \right) \xi^2 + \int_{[-m, m]^c} (1 - \cos \xi y) \nu_1(x, dy), \end{aligned}$$

where in the fourth step we applied the fact that $1 - \cos y \leq y^2$ for all $y \in \mathbb{R}$. Thus,

$$\sup_{x \in \mathbb{R}} q_2(x, \xi) \leq c\xi^2 + \sup_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_1(x, dy),$$

where

$$c = \frac{1}{2} \sup_{x \in \mathbb{R}} c_2(x) + \sup_{x \in \mathbb{R}} \int_{[-m, m]} y^2 \nu_2(x, dy).$$

By the same reasoning, we get

$$\sup_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_1(x, dy) \leq \sup_{x \in \mathbb{R}} q_1(x, \xi) \leq \bar{c} \xi^2 + \sup_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_1(x, dy),$$

for some $\bar{c} > 0$. Next, (3.3) implies

$$\lim_{\xi \rightarrow 0} \frac{\sup_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_1(x, dy)}{\xi^2} = \infty,$$

$$\lim_{\xi \rightarrow 0} \frac{\sup_{x \in \mathbb{R}} q_1(x, \xi)}{\sup_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_1(x, dy)} = 1$$

and

$$\lim_{\xi \rightarrow 0} \frac{c \xi^2 + \sup_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_1(x, dy)}{\sup_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_1(x, dy)} = 1.$$

Therefore, by applying Proposition 2.4, the claim follows.

Now, we prove the transience case. Again, let $m > 0$ be so large that $C \subseteq [-m, m]$. Clearly,

$$\inf_{x \in \mathbb{R}} q_1(x, \xi) \geq \inf_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_1(x, dy) \geq \inf_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_2(x, dy)$$

and

$$\begin{aligned} q_2(x, \xi) &= \frac{1}{2} c_2(x) \xi^2 + \int_{\mathbb{R}} (1 - \cos \xi y) \nu_2(x, dy) \\ &= \frac{1}{2} c_2(x) \xi^2 + \int_{[-m, m]} (1 - \cos \xi y) \nu_2(x, dy) + \int_{[-m, m]^c} (1 - \cos \xi y) \nu_2(x, dy) \\ &\leq \frac{1}{2} \sup_{x \in \mathbb{R}} c_2(x) \xi^2 + \xi^2 \sup_{x \in \mathbb{R}} \int_{[-m, m]} y^2 \nu_2(x, dy) + \sup_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_2(x, dy) \\ &\leq \xi^2 \left(\frac{1}{2} \sup_{x \in \mathbb{R}} c_2(x) + \sup_{x \in \mathbb{R}} \int_{[-m, m]} y^2 \nu_2(x, dy) \right) + \sup_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_2(x, dy). \end{aligned}$$

Thus,

$$\inf_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_2(x, dy) \leq \inf_{x \in \mathbb{R}} q_2(x, \xi) \leq c \xi^2 + \inf_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_2(x, dy),$$

where

$$c = \frac{1}{2} \sup_{x \in \mathbb{R}} c_2(x) + \sup_{x \in \mathbb{R}} \int_{[-m, m]} y^2 \nu_2(x, dy).$$

Now, by (3.2), we get

$$\lim_{\xi \rightarrow \infty} \frac{\inf_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_2(x, dy)}{\xi^2} = \infty,$$

$$\lim_{\xi \rightarrow \infty} \frac{\inf_{x \in \mathbb{R}} q_2(x, \xi)}{\inf_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_2(x, dy)} = 1$$

and

$$\lim_{\xi \rightarrow \infty} \frac{c \xi^2 + \inf_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_2(x, dy)}{\inf_{x \in \mathbb{R}} \int_{[-m, m]^c} (1 - \cos \xi y) \nu_2(x, dy)} = 1.$$

Now, the desired result follows from Proposition 2.4. \square

In many situations the Chung-Fuchs type conditions in (1.3) and (1.4) are not operable. More precisely, it is not always easy to compute the integrals appearing in (1.3) and (1.4). According to this, in the sequel we derive necessary and sufficient conditions for the recurrence and transience of one-dimensional symmetric nice Feller processes in terms of the Lévy measure. First, recall that a symmetric Borel measure $\mu(dy)$ on $\mathcal{B}(\mathbb{R})$ is *quasi-unimodal* if there exists $y_0 \geq 0$ such that $y \mapsto \mu(y, \infty)$ is a convex function on (y_0, ∞) . Equivalently, a symmetric Borel measure $\mu(dy)$ on $\mathcal{B}(\mathbb{R})$ is quasi-unimodal if it is of the form $\mu(dy) = \mu_0(dy) + f(y)dy$, where the measure $\mu_0(dy)$ is supported on $[-y_0, y_0]$, for some $y_0 \geq 0$, and the density function $f(y)$ is supported on $[-y_0, y_0]^c$, it is symmetric and decreasing on (y_0, ∞) and $\int_{y_0+\varepsilon}^{\infty} f(y)dy < \infty$ for every $\varepsilon > 0$ (see [23, Chapters 5 and 7]). When $y_0 = 0$, then $\mu(dy)$ is said to be *unimodal*.

Theorem 3.5. *Let $\{F_t\}_{t \geq 0}$ be a one-dimensional symmetric nice Feller process with symbol $q(x, \xi)$ and Lévy measure $\nu(x, dy)$. Assume that there exists $x_0 \in \mathbb{R}$ such that*

- (i) $\inf_{x \in \mathbb{R}} q(x, \xi) = q(x_0, \xi)$ for all $|\xi|$ small enough
- (ii) the Lévy measure $\nu(x_0, dy)$ is quasi-unimodal
- (iii) there exists a one-dimensional symmetric Lévy process $\{L_t\}_{t \geq 0}$ with symbol $q(\xi)$ and Lévy measure $\nu(dy)$, such that $\nu(x_0, dy)$ has a bigger tail than $\nu(dy)$.

Then, the transience property of $\{L_t\}_{t \geq 0}$ implies (1.4).

Proof. By Theorem 1.3, it suffices to prove that

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\inf_{x \in \mathbb{R}} q(x, \xi)} = \int_{\{|\xi| < r\}} \frac{d\xi}{q(x_0, \xi)} < \infty$$

for some $r > 0$. Let $\{F_t^0\}_{t \geq 0}$ be a Lévy process with symbol $q(x_0, \xi)$. Now, by [23, Theorem 38.2], $\{F_t^0\}_{t \geq 0}$ is transient. Hence, by [23, Theorem 37.5],

$$\int_{\{|\xi| < r\}} \frac{d\xi}{q(x_0, \xi)} < \infty$$

for all $r > 0$. □

As a direct consequence of the above result we get the following corollary.

Corollary 3.6. *Let $\{F_t\}_{t \geq 0}$ be a one-dimensional symmetric nice Feller process with symbol $q(x, \xi)$ and Lévy measure $\nu(x, dy)$. Assume that there exists $x_0 \in \mathbb{R}$ such that*

- (i) $\sup_{x \in \mathbb{R}} q(x, \xi) = q(x_0, \xi)$ for all $|\xi|$ small enough
- (ii) there exists a one-dimensional symmetric Lévy process $\{L_t\}_{t \geq 0}$ with symbol $q(\xi)$ and Lévy measure $\nu(dy)$, such that $\nu(dy)$ is quasi-unimodal and has a bigger tail than $\nu(x_0, dy)$.

Then, the recurrence property of $\{L_t\}_{t \geq 0}$ implies (1.3).

For the necessity of the quasi-unimodality assumption in Theorem 3.5 and Corollary 3.6 see [23, Theorem 38.4]. Explicit examples of one-dimensional symmetric nice Feller processes which satisfy the conditions in Theorem 3.5 and Corollary 3.6 can be easily constructed in the classes of stable-like processes and Feller processes obtained by variable order subordination (see Section 2).

Theorem 3.7. Let $\{F_t\}_{t \geq 0}$ be a one-dimensional symmetric nice Feller process with symbol

$$q(x, \xi) = \frac{1}{2}c(x)\xi^2 + 2 \int_0^\infty (1 - \cos \xi y) \nu(x, dy).$$

Let us define

$$R(x, r, y) := \nu \left(x, \bigcup_{n=0}^\infty (2nr + y, 2(n+1)r - y] \right),$$

for $x \in \mathbb{R}$ and $r \geq y \geq 0$. Then, for arbitrary $\rho > 0$,

$$\int_\rho^\infty \left(\sup_{x \in \mathbb{R}} \int_0^r y R(x, r, y) dy \right)^{-1} dr = \infty \quad (3.6)$$

if, and only if, (1.3) holds true. Further, under (3.2), (1.4) holds true if, and only if,

$$\int_\rho^\infty \left(\inf_{x \in \mathbb{R}} \int_0^r y R(x, r, y) dy \right)^{-1} dr < \infty \quad (3.7)$$

holds for all $\rho > 0$ large enough.

Proof. We follow the proof of [23, Theorem 38.3]. Let us denote $N(x, y) := \nu(x, (y, \infty))$, for $x \in \mathbb{R}$ and $y \geq 0$. Then, we have

$$\begin{aligned} q(x, \xi) - \frac{1}{2}c(x)\xi^2 &= 2 \int_0^\infty (1 - \cos \xi y) \nu(x, dy) \\ &= 2 \int_0^\infty (1 - \cos \xi y) d(-N(x, y)) \\ &= 2\xi \int_0^\infty N(x, y) \sin \xi y dy \\ &= 2\xi \sum_{n=0}^\infty \int_0^{\frac{2\pi}{\xi}} N \left(x, \frac{2\pi n}{\xi} + y \right) \sin \xi y dy \\ &= 2\xi \sum_{n=0}^\infty (I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}), \end{aligned}$$

where in the third step we applied the integration by parts formula and in the final step we wrote

$$\begin{aligned} I_{n,1} &= \int_0^{\frac{\pi}{2\xi}} N \left(x, \frac{2\pi n}{\xi} + y \right) \sin \xi y dy, \\ I_{n,2} &= \int_{\frac{\pi}{2\xi}}^{\frac{\pi}{\xi}} N \left(x, \frac{2\pi n}{\xi} + y \right) \sin \xi y dy = \int_0^{\frac{\pi}{2\xi}} N \left(x, \frac{2\pi n}{\xi} + \frac{\pi}{\xi} - y \right) \sin \xi y dy, \\ I_{n,3} &= \int_{\frac{\pi}{\xi}}^{\frac{3\pi}{2\xi}} N \left(x, \frac{2\pi n}{\xi} + y \right) \sin \xi y dy = - \int_0^{\frac{\pi}{2\xi}} N \left(x, \frac{2\pi n}{\xi} + \frac{\pi}{\xi} + y \right) \sin \xi y dy \end{aligned}$$

and

$$I_{n,4} = \int_{\frac{3\pi}{2\xi}}^{\frac{2\pi}{\xi}} N \left(x, \frac{2\pi n}{\xi} + y \right) \sin \xi y dy = - \int_0^{\frac{\pi}{2\xi}} N \left(x, \frac{2\pi n}{\xi} + \frac{2\pi}{\xi} - y \right) \sin \xi y dy.$$

Thus,

$$I_{n,1} + I_{n,4} = \int_0^{\frac{\pi}{2\xi}} \nu \left(x, \left(\frac{2\pi n}{\xi} + y, \frac{2\pi(n+1)}{\xi} - y \right) \right) \sin \xi y \, dy$$

and

$$I_{n,2} + I_{n,3} = \int_0^{\frac{\pi}{2\xi}} \nu \left(x, \left(\frac{\pi(2n+1)}{\xi} - y, \frac{\pi(2n+1)}{\xi} + y \right) \right) \sin \xi y \, dy.$$

Now, by defining

$$\bar{R}(x, r, y) := \nu \left(x, \bigcup_{n=0}^{\infty} ((2n+1)r - y, (2n+1)r + y) \right),$$

we have

$$q(x, \xi) - \frac{1}{2}c(x)\xi^2 = 2\xi \left(\int_0^{\frac{\pi}{2\xi}} R \left(x, \frac{\pi}{\xi}, y \right) \sin \xi y \, dy + \int_0^{\frac{\pi}{2\xi}} \bar{R} \left(x, \frac{\pi}{\xi}, y \right) \sin \xi y \, dy \right).$$

Further, note that

$$R \left(x, \frac{\pi}{\xi}, y \right) \geq \bar{R} \left(x, \frac{\pi}{\xi}, y \right) \geq 0, \quad y \in \left(0, \frac{\pi}{2\xi} \right],$$

and

$$\frac{2y}{\pi} \leq \sin y \leq y, \quad y \in \left(0, \frac{\pi}{2} \right].$$

This yields

$$\frac{4}{\pi}\xi^2 \int_0^{\frac{\pi}{2\xi}} y R \left(x, \frac{\pi}{\xi}, y \right) dy \leq q(x, \xi) - \frac{1}{2}c(x)\xi^2 \leq 4\xi^2 \int_0^{\frac{\pi}{2\xi}} y R \left(x, \frac{\pi}{\xi}, y \right) dy. \quad (3.8)$$

Next, we have

$$\int_0^{\frac{\pi}{\rho}} \left(\xi^2 \sup_{x \in \mathbb{R}} \int_0^{\frac{\pi}{\xi}} y R \left(x, \frac{\pi}{\xi}, y \right) dy \right)^{-1} d\xi = \frac{1}{\pi} \int_{\rho}^{\infty} \left(\sup_{x \in \mathbb{R}} \int_0^r y R(x, r, y) dy \right)^{-1} dr$$

and

$$\int_0^{\frac{\pi}{\rho}} \left(\xi^2 \inf_{x \in \mathbb{R}} \int_0^{\frac{\pi}{\xi}} y R \left(x, \frac{\pi}{\xi}, y \right) dy \right)^{-1} d\xi = \frac{1}{\pi} \int_{\rho}^{\infty} \left(\inf_{x \in \mathbb{R}} \int_0^r y R(x, r, y) dy \right)^{-1} dr,$$

where we made the substitution $\xi \mapsto \pi/r$. Thus, (3.6) implies

$$\int_0^{\frac{\pi}{\rho}} \frac{d\xi}{\sup_{x \in \mathbb{R}} (q(x, \xi) - \frac{1}{2}c(x)\xi^2)} = \infty$$

and

$$\int_0^{\frac{\pi}{\rho}} \frac{d\xi}{\inf_{x \in \mathbb{R}} (q(x, \xi) - \frac{1}{2}c(x)\xi^2)} < \infty$$

implies (3.7). Finally, from (3.2) and (3.3), we have

$$\lim_{\xi \rightarrow 0} \frac{\sup_{x \in \mathbb{R}} q(x, \xi)}{\sup_{x \in \mathbb{R}} (q(x, \xi) - \frac{1}{2}c(x)\xi^2)} = \lim_{\xi \rightarrow 0} \frac{\inf_{x \in \mathbb{R}} q(x, \xi)}{\inf_{x \in \mathbb{R}} (q(x, \xi) - \frac{1}{2}c(x)\xi^2)} = 1. \quad (3.9)$$

Now, the claim follows from Proposition 2.4.

To prove the converse, first note that

$$\begin{aligned}
& \int_0^{\frac{\pi}{\xi}} y \nu \left(x, \left(\frac{2n\pi}{\xi} + y, \frac{2(n+1)\pi}{\xi} - y \right) \right) dy \\
&= \int_0^{\frac{\pi}{2\xi}} y \nu \left(x, \left(\frac{2n\pi}{\xi} + y, \frac{2(n+1)\pi}{\xi} - y \right) \right) dy + \int_{\frac{\pi}{2\xi}}^{\frac{\pi}{\xi}} y \nu \left(x, \left(\frac{2n\pi}{\xi} + y, \frac{2(n+1)\pi}{\xi} - y \right) \right) dy \\
&= \int_0^{\frac{\pi}{2\xi}} y \nu \left(x, \left(\frac{2n\pi}{\xi} + y, \frac{2(n+1)\pi}{\xi} - y \right) \right) dy + 4 \int_{\frac{\pi}{4\xi}}^{\frac{\pi}{2\xi}} y \nu \left(x, \left(\frac{2n\pi}{\xi} + 2y, \frac{2(n+1)\pi}{\xi} - 2y \right) \right) dy \\
&\leq \int_0^{\frac{\pi}{2\xi}} y \nu \left(x, \left(\frac{2n\pi}{\xi} + y, \frac{2(n+1)\pi}{\xi} - y \right) \right) dy + 4 \int_{\frac{\pi}{4\xi}}^{\frac{\pi}{2\xi}} y \nu \left(x, \left(\frac{2n\pi}{\xi} + y, \frac{2(n+1)\pi}{\xi} - y \right) \right) dy \\
&\leq 5 \int_0^{\frac{\pi}{2\xi}} y \nu \left(x, \left(\frac{2n\pi}{\xi} + y, \frac{2(n+1)\pi}{\xi} - y \right) \right) dy.
\end{aligned}$$

Hence,

$$\int_0^{\frac{\pi}{\xi}} y R \left(x, \frac{\pi}{\xi}, y \right) dy \leq 5 \int_0^{\frac{\pi}{2\xi}} y R \left(x, \frac{\pi}{\xi}, y \right) dy,$$

that is,

$$\begin{aligned}
\int_{\rho}^{\infty} \left(\sup_{x \in \mathbb{R}} \int_0^r y R(x, r, y) dy \right)^{-1} dr &= \int_0^{\frac{\pi}{\rho}} \left(\xi^2 \sup_{x \in \mathbb{R}} \int_0^{\frac{\pi}{\xi}} y R \left(x, \frac{\pi}{\xi}, y \right) dy \right)^{-1} d\xi \\
&\geq \frac{1}{5} \int_0^{\frac{\pi}{\rho}} \left(\xi^2 \sup_{x \in \mathbb{R}} \int_0^{\frac{\pi}{2\xi}} y R \left(x, \frac{\pi}{\xi}, y \right) dy \right)^{-1} d\xi
\end{aligned}$$

and

$$\begin{aligned}
\int_{\rho}^{\infty} \left(\inf_{x \in \mathbb{R}} \int_0^r y R(x, r, y) dy \right)^{-1} dr &= \int_0^{\frac{\pi}{\rho}} \left(\xi^2 \inf_{x \in \mathbb{R}} \int_0^{\frac{\pi}{\xi}} y R \left(x, \frac{\pi}{\xi}, y \right) dy \right)^{-1} d\xi \\
&\geq \frac{1}{5} \int_0^{\frac{\pi}{\rho}} \left(\xi^2 \inf_{x \in \mathbb{R}} \int_0^{\frac{\pi}{2\xi}} y R \left(x, \frac{\pi}{\xi}, y \right) dy \right)^{-1} d\xi,
\end{aligned}$$

where in the first steps we applied the substitution $\xi \mapsto \pi/r$. Thus, (1.3) and (3.7), by using (3.2), (3.3), (3.8), (3.9) and Proposition 2.4, imply (3.6) and (1.4), respectively. \square

As a consequence of Theorem 3.7, we also get the following characterization of the recurrence and transience in terms of the tail behavior of the Lévy measure.

Corollary 3.8. *Let $\{F_t\}_{t \geq 0}$ be a one-dimensional symmetric nice Feller process with symbol $q(x, \xi)$ and Lévy measure $\nu(x, dy)$. Let us define*

$$N(x, y) := \nu(x, (y, \infty)),$$

for $x \in \mathbb{R}$ and $y \geq 0$. Then, for arbitrary $\rho > 0$,

$$\int_{\rho}^{\infty} \left(\sup_{x \in \mathbb{R}} \int_0^r y N(x, y) dy \right)^{-1} dr = \infty \tag{3.10}$$

implies (1.3), and (1.4) implies

$$\int_{\rho}^{\infty} \left(\inf_{x \in \mathbb{R}} \int_0^r y N(x, y) dy \right)^{-1} dr < \infty \quad (3.11)$$

for all $\rho > 0$ large enough.

Proof. The claim directly follows from the fact $N(x, y) \geq R(x, r, y)$ for all $x \in \mathbb{R}$ and all $0 \leq y \leq r$. \square

In addition, if we assume the quasi-unimodality of the Lévy measure $\nu(x, dy)$, then we can prove the equivalence in Corollary 3.8.

Theorem 3.9. *Let $\{F_t\}_{t \geq 0}$ be a one-dimensional symmetric nice Feller process with symbol $q(x, \xi)$ and Lévy measure $\nu(x, dy)$, such that*

(i) $\nu(x, dy)$ is quasi-unimodal uniformly in $x \in \mathbb{R}$

(ii) the function $x \mapsto \nu(x, O - x)$ is lower semicontinuous for every open set $O \subseteq \mathbb{R}$, that is, $\liminf_{y \rightarrow x} \nu(y, O - y) \geq \nu(x, O - x)$ for all $x \in \mathbb{R}$ and all open sets $O \subseteq \mathbb{R}$.

Then, (3.10) holds true if, and only if, (1.3) holds true. Further, if (3.2) and

$$\inf_{x \in \mathbb{R}} \int_{y_0}^{\infty} y \nu(x, (y, \infty)) dy > 0 \quad (3.12)$$

hold true for some $y_0 > 0$, then (3.11) holds true if, and only if, (1.4) holds true.

Proof. The proof is divided in three steps.

Step 1. In the first step, we construct a nice Feller processes $\{\bar{F}_t\}_{t \geq 0}$ with finite and unimodal Lévy measure which has the same tails as $\nu(x, dy)$. Then, in particular, by Theorem 3.1, the conditions in (1.3) and (1.4) are equivalent for $\{\bar{F}_t\}_{t \geq 0}$ and $\{F_t\}_{t \geq 0}$. By assumptions (i) and (C2), there exists $y_0 > 1$ such that $\sup_{x \in \mathbb{R}} \nu(x, (y_0 - 1, \infty)) < \infty$ and $y \mapsto \nu(x, (y, \infty))$ is convex on $(y_0 - 1, \infty)$ for all $x \in \mathbb{R}$. Next, let $f(x, y)$ be the density of $\nu(x, dy)$ on $(y_0 - 1, \infty)$, that is, $\frac{\partial}{\partial y} \nu(x, (y, \infty)) = -f(x, y)$ on $(y_0 - 1, \infty)$. Further, note that $\sup_{x \in \mathbb{R}} f(x, y_0) < \infty$. Indeed, if this was not the case, then we would have

$$\sup_{x \in \mathbb{R}} \nu(x, (y_0 - 1, \infty)) = \sup_{x \in \mathbb{R}} \int_{y_0 - 1}^{\infty} f(x, y) dy \geq \sup_{x \in \mathbb{R}} f(x, y_0) \int_{y_0 - 1}^{y_0} dy = \sup_{x \in \mathbb{R}} f(x, y_0) = \infty,$$

where in the second step we employed the fact that $y \mapsto f(x, y)$ is decreasing on $(y_0 - 1, \infty)$ for all $x \in \mathbb{R}$. Let $c := y_0 \sup_{x \in \mathbb{R}} f(x, y_0) + \sup_{x \in \mathbb{R}} \nu(x, (y_0, \infty)) + 1$ and let $\bar{\nu}(x, dy)$ be a symmetric probability kernel on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by $\bar{\nu}(x, \{0\}) = 0$ and

$$\bar{\nu}(x, (y, \infty)) := \begin{cases} \frac{\nu(x, (y_0, \infty)) - c}{2cy_0} \lambda(0, y) + \frac{1}{2}, & 0 < y \leq y_0 \\ \frac{\nu(x, (y, \infty))}{2c}, & y \geq y_0, \end{cases}$$

for all $x \in \mathbb{R}$. Note that, since

$$\frac{\partial}{\partial y} \bar{\nu}(x, (y, \infty)) \Big|_{y_0} = -\frac{f(x, y_0)}{2c} \quad \text{and} \quad \frac{\partial}{\partial y} \bar{\nu}(x, (y, \infty)) = \frac{\nu(x, (y_0, \infty)) - c}{2cy_0}$$

for $y \in (0, y_0)$, $\bar{\nu}(x, dy)$ is unimodal. Next, put $\bar{p}(x, dy) := \bar{\nu}(x, dy - x)$. Clearly, $\bar{p}(x, dy)$ is a probability kernel on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which defines a Markov chain, say $\{F_n\}_{n \geq 0}$. Further, let $\{P_t\}_{t \geq 0}$ be the Poisson process with intensity $\lambda = 2c$ independent of $\{F_n\}_{n \geq 0}$. Then, by $\bar{F}_t := F_{P_t}$, $t \geq 0$, is well defined a Markov process with the transition kernel

$$\mathbb{P}^x(\bar{F}_t \in dy) = e^{-2ct} \sum_{n=0}^{\infty} \frac{(2ct)^n}{n!} \bar{p}^n(x, dy),$$

here $\bar{p}^0(x, dy)$ is the Dirac measure $\delta_x(dy)$ and

$$\bar{p}^n(x, dy) := \int_{\mathbb{R}} \dots \int_{\mathbb{R}} \bar{p}(x, dy_1) \dots \bar{p}(y_{n-1}, dy),$$

for $n \geq 1$. Note that $\{\bar{F}_t\}_{t \geq 0}$ is a nice Feller process with symbol $\bar{q}(x, \xi) = 2c \int_{\mathbb{R}} (1 - \cos \xi y) \bar{\nu}(x, dy)$. Indeed, the strong continuity property can be easily verified. Next, in order to prove the continuity property of $x \mapsto \int_{\mathbb{R}} \mathbb{P}^x(\bar{F}_t \in dy) f(y)$, for $f \in C_b(\mathbb{R})$ and $t \geq 0$, by [14, Proposition 6.1.1], it suffices to show the lower semicontinuity property of the function $x \mapsto \bar{\nu}(x, O - x)$ for all open sets $O \subseteq \mathbb{R}$. But this is the assumption (ii). Finally, we show that the function $x \mapsto \int_{\mathbb{R}} \mathbb{P}^x(\bar{F}_t \in dy) f(y)$ vanishes at infinity for all $f \in C_{\infty}(\mathbb{R})$ and all $t \geq 0$. Let $f \in C_{\infty}(\mathbb{R})$ and $\varepsilon > 0$ be arbitrary and let $m > 0$ be such that $\|f\|_{\infty} \leq m$. Since $C_c(\mathbb{R})$ is dense in $(C_{\infty}(\mathbb{R}), \|\cdot\|_{\infty})$, there exists $f_{\varepsilon} \in C_c(\mathbb{R})$ such that $\|f - f_{\varepsilon}\|_{\infty} < \varepsilon$. We have

$$\begin{aligned} \left| \int_{\mathbb{R}} \bar{p}(x, dy) f(y) \right| &\leq \int_{\mathbb{R}} \bar{p}(x, dy) |f(y)| \\ &< \int_{\mathbb{R}} \bar{p}(x, dy) |f_{\varepsilon}(y)| + \varepsilon \\ &= \int_{\mathbb{R}} \bar{\nu}(x, dy) |f_{\varepsilon}(y + x)| dy + \varepsilon \\ &\leq (m + \varepsilon) \int_{\text{supp } f_{\varepsilon} - x} \bar{\nu}(x, dy) + \varepsilon \\ &= (m + \varepsilon) \bar{\nu}(x, \text{supp } f_{\varepsilon} - x) + \varepsilon. \end{aligned}$$

Now, since $\text{supp } f_{\varepsilon} := \{y : f_{\varepsilon}(y) \neq 0\}$ has compact closure, it suffices to prove that $\lim_{|x| \rightarrow \infty} \bar{\nu}(x, C - x) = 0$ for every compact set $C \subseteq \mathbb{R}$. Let $C \subseteq \mathbb{R}$ be a compact set. Then, for arbitrary $r > 0$ and $|x|$ large enough, we have

$$\bar{\nu}(x, C - x) = \frac{\nu(x, C - x)}{2c} \leq \frac{\nu(x, (-r, r)^c)}{2c} \leq \frac{\sup_{x \in \mathbb{R}} \nu(x, (-r, r)^c)}{2c}.$$

Hence,

$$\limsup_{|x| \rightarrow \infty} \bar{\nu}(x, C - x) \leq \frac{\sup_{x \in \mathbb{R}} \nu(x, (-r, r)^c)}{2c}.$$

Now, by letting $r \rightarrow \infty$, from [24, Theorem 4.4], we get the desired result. Finally, it can be easily verified that the symbol of $\{\bar{F}_t\}_{t \geq 0}$ is given by $\bar{q}(x, \xi) = 2c \int_{\mathbb{R}} (1 - \cos \xi y) \bar{\nu}(x, dy)$ and obviously, by the definition, $\{\bar{F}_t\}_{t \geq 0}$ satisfies conditions (C1)-(C4).

Step 2. In Corollary 3.8 we have proved that (3.10) implies (1.3). In the second step, we prove the converse. Since $\bar{\nu}(x, dy)$ is unimodal, by [23, Exercise 29.21], there exists a random variable F_x such that $\bar{\nu}(x, dy)$ is the distribution of the random variable UF_x , where U is uniformly distributed random variable on $[0, 1]$ independent of F_x . Further, let $\bar{\nu}_U(x, dy)$ be the distribution

of the random variable F_x . By [23, Lemma 38.6], $\bar{\nu}_U(x, (y, \infty)) \geq \bar{\nu}(x, (y, \infty))$ for all $x \in \mathbb{R}$ and all $y \geq 0$. Now, we have

$$\begin{aligned}\bar{q}(x, \xi) &= 2c \int_{\mathbb{R}} (1 - \cos \xi y) \bar{\nu}(x, dy) \\ &= 2c \int_0^1 \int_{\mathbb{R}} (1 - \cos(\xi u y)) \bar{\nu}_U(x, dy) du \\ &= 2c \int_{\mathbb{R}} \left(1 - \frac{\sin \xi y}{\xi y}\right) \bar{\nu}_U(x, dy).\end{aligned}$$

Further, since

$$1 - \frac{\sin y}{y} \geq \bar{c} \min\{1, y^2\}$$

for all $y \in \mathbb{R}$ and all $0 < \bar{c} < \frac{1}{6}$, we have

$$\bar{q}(x, \xi) \geq 2c\bar{c} \int_{\mathbb{R}} \min\{1, (\xi y)^2\} \bar{\nu}_U(x, dy) = 8c\bar{c}\xi^2 \int_0^{\frac{1}{|\xi|}} y \bar{N}_U(x, y) dy,$$

where $\bar{N}_U(x, y) := \bar{\nu}_U(x, (y, \infty))$, for $x \in \mathbb{R}$ and $y \geq 0$. Finally, let us put $\bar{N}(x, y) := \bar{\nu}(x, (y, \infty))$, for $x \in \mathbb{R}$ and $y \geq 0$, then we have

$$\begin{aligned}\int_{\rho}^{\infty} \left(\sup_{x \in \mathbb{R}} \int_0^r y \bar{N}(x, y) dy \right)^{-1} dr &\geq \int_{\rho}^{\infty} \left(\sup_{x \in \mathbb{R}} \int_0^r y \bar{N}_U(x, y) dy \right)^{-1} dr \\ &= \int_{\{|\xi| < \frac{1}{\rho}\}} \left(\xi^2 \sup_{x \in \mathbb{R}} \int_0^{\frac{1}{|\xi|}} y \bar{N}_U(x, y) dy \right)^{-1} d\xi \\ &\geq 8c\bar{c} \int_{\{|\xi| < \frac{1}{\rho}\}} \frac{d\xi}{\sup_{x \in \mathbb{R}} \bar{q}(x, \xi)}.\end{aligned}$$

Further, we have

$$\begin{aligned}&\lim_{r \rightarrow \infty} \frac{\sup_{x \in \mathbb{R}} \int_{y_0}^r y N(x, y) dy}{\sup_{x \in \mathbb{R}} \int_0^{y_0} y \bar{N}(x, y) dy + \frac{1}{2c} \sup_{x \in \mathbb{R}} \int_{y_0}^r y N(x, y) dy} \\ &\leq \liminf_{r \rightarrow \infty} \frac{\sup_{x \in \mathbb{R}} \int_0^r y N(x, y) dy}{\sup_{x \in \mathbb{R}} \int_0^r y \bar{N}(x, y) dy} \\ &\leq \limsup_{r \rightarrow \infty} \frac{\sup_{x \in \mathbb{R}} \int_0^r y N(x, y) dy}{\sup_{x \in \mathbb{R}} \int_0^r y \bar{N}(x, y) dy} \\ &\leq \lim_{r \rightarrow \infty} \frac{\sup_{x \in \mathbb{R}} \int_0^{y_0} y N(x, y) dy + \sup_{x \in \mathbb{R}} \int_{y_0}^r y N(x, y) dy}{\frac{1}{2c} \sup_{x \in \mathbb{R}} \int_{y_0}^r y N(x, y) dy}\end{aligned}$$

Now, if $\sup_{x \in \mathbb{R}} \int_{y_0}^{\infty} y N(x, y) dy = 0$ the claim trivially follows. On the other hand, if $\sup_{x \in \mathbb{R}} \int_{y_0}^{\infty} y N(x, y) dy > 0$, we have

$$0 < \liminf_{r \rightarrow \infty} \frac{\sup_{x \in \mathbb{R}} \int_0^r y N(x, y) dy}{\sup_{x \in \mathbb{R}} \int_0^r y \bar{N}(x, y) dy} \leq \limsup_{r \rightarrow \infty} \frac{\sup_{x \in \mathbb{R}} \int_0^r y N(x, y) dy}{\sup_{x \in \mathbb{R}} \int_0^r y \bar{N}(x, y) dy} < \infty, \quad (3.13)$$

which together with Proposition 2.4 and Theorem 3.1 proves the claim.

Step 3. In the third step, we consider the second part of the theorem. In Corollary 3.8 we have proved that (1.4) implies (3.11). Now, to prove the converse, by completely the same arguments as in the second step, we have

$$8c\bar{c} \int_{\{|\xi| < \frac{1}{\rho}\}} \frac{d\xi}{\inf_{x \in \mathbb{R}} \bar{q}(x, \xi)} \leq \int_{\rho}^{\infty} \left(\inf_{x \in \mathbb{R}} \int_0^r y \bar{N}(x, y) dy \right)^{-1} dr.$$

Now, the claim follows from Proposition 2.4 (ii), (3.2), Theorem 3.1, (3.12) and a similar argumentation as in (3.13). \square

In the sequel we discuss some consequences of Theorem 3.9.

Theorem 3.10. *Let $\{F_t\}_{t \geq 0}$ be a one-dimensional symmetric nice Feller process with symbol*

$$q(x, \xi) = \frac{1}{2}c(x)\xi^2 + 2 \int_0^{\infty} (1 - \cos \xi y) \nu(x, dy)$$

satisfying (3.2). Then, for any $r > 0$, and any $0 < a < b < \pi$ fixed, either one of the following conditions

$$\int_r^{\infty} \frac{dy}{y^2 \inf_{x \in \mathbb{R}} (N(x, by) - N(x, (a + \pi)y))} < \infty, \quad (3.14)$$

or

$$\int_r^{\infty} \frac{d\xi}{\inf_{x \in \mathbb{R}} \int_0^{\xi} y^2 \nu(x, dy)} < \infty \quad (3.15)$$

implies (1.4). So that if (3.14) or (3.15) is satisfied, then the process $\{F_t\}_{t \geq 0}$ is transient.

Proof. Let $N(x, y) := \nu(x, (y, \infty))$, for $x \in \mathbb{R}$ and $y \geq 0$. Now, by the integration by parts formula, we have

$$\begin{aligned} q(x, \xi) - \frac{1}{2}c(x)\xi^2 &= 2 \int_0^{\infty} (1 - \cos \xi y) \nu(x, dy) \\ &= 2 \int_0^{\infty} (1 - \cos \xi y) d(-N(x, y)) \\ &= 2\xi \int_0^{\infty} \sin \xi y N(x, y) dy \\ &= 2 \int_0^{\infty} \sin y N\left(x, \frac{y}{\xi}\right) dy \\ &\geq 2 \int_0^{2\pi} \sin y N\left(x, \frac{y}{\xi}\right) dy \\ &= 2 \int_0^{\pi} \sin y \left(N\left(x, \frac{y}{\xi}\right) - N\left(x, \frac{y + \pi}{\xi}\right) \right) dy, \end{aligned}$$

where in the last two inequalities we used the periodicity of the sine function and the nonincreasing property of $y \mapsto N(x, y)$. Therefore, for any $0 < a < b < \pi$, we have that

$$q(x, \xi) - \frac{1}{2}c(x)\xi^2 \geq 2 \int_a^b \sin y \left(N\left(x, \frac{y}{\xi}\right) - N\left(x, \frac{y + \pi}{\xi}\right) \right) dy,$$

and hence

$$q(x, \xi) - \frac{1}{2}c(x)\xi^2 \geq c(a, b) \left(N\left(x, \frac{b}{\xi}\right) - N\left(x, \frac{a+\pi}{\xi}\right) \right),$$

where $c(a, b) := 2(b - a) \inf_{y \in (a, b)} \sin y$. This yields

$$\begin{aligned} \int_{\{|\xi| < \frac{1}{r}\}} \frac{d\xi}{\inf_{x \in \mathbb{R}} (q(x, \xi) - \frac{1}{2}c(x)\xi^2)} &\leq c^{-1}(a, b) \int_0^{\frac{1}{r}} \frac{d\xi}{\inf_{x \in \mathbb{R}} \left(N\left(x, \frac{b}{\xi}\right) - N\left(x, \frac{a+\pi}{\xi}\right) \right)} \\ &= c^{-1}(a, b) \int_r^\infty \frac{dy}{y^2 \inf_{x \in \mathbb{R}} (N(x, by) - N(x, (a+\pi)y))}, \end{aligned}$$

which together with Proposition 2.4 and Theorem 3.1 proves the desired result.

To prove the second claim, first note that

$$q(x, \xi) - \frac{1}{2}c(x)\xi^2 = 2 \int_0^\infty (1 - \cos \xi y) \nu(x, dy) \geq \frac{2}{\pi} \xi^2 \int_0^{\frac{1}{|\xi|}} y^2 \nu(x, dy),$$

where we applied the fact that $1 - \cos y \geq \frac{1}{\pi} y^2$ for all $|y| \leq \frac{\pi}{2}$. This yields

$$\begin{aligned} \int_{\{|\xi| < \frac{1}{r}\}} \frac{d\xi}{\inf_{x \in \mathbb{R}} (q(x, \xi) - \frac{1}{2}c(x)\xi^2)} &\leq \frac{\pi}{2} \int_{\{|\xi| < \frac{1}{r}\}} \frac{d\xi}{\xi^2 \inf_{x \in \mathbb{R}} \int_0^{\frac{1}{|\xi|}} y^2 \nu(x, dy)} \\ &= \pi \int_r^\infty \frac{d\xi}{\inf_{x \in \mathbb{R}} \int_0^\xi y^2 \nu(x, dy)}, \end{aligned}$$

where in the second step we made the substitution $\xi \rightarrow \xi^{-1}$. Now, the desired result is an immediate consequence of Proposition 2.4 and Theorem 3.1. \square

In addition, by assuming the quasi-unimodality property of the Lévy measure we get the following sufficient condition for transience.

Theorem 3.11. *Let $\{F_t\}_{t \geq 0}$ be a one-dimensional symmetric nice Feller process with symbol $q(x, \xi)$ and Lévy measure $\nu(x, dy)$ satisfying the assumptions from Theorem 3.9. Further, let $y_0 > 0$ be a constant of uniform quasi-unimodality of $\nu(x, dy)$. Then,*

$$\int_{y_0}^\infty \frac{dy}{y^3 \inf_{x \in \mathbb{R}} f(x, y)} < \infty$$

implies (1.4).

Proof. By Theorem 3.9, it suffices to show that

$$\int_{y_0}^\infty \left(\inf_{x \in \mathbb{R}} \int_0^r y N(x, y) dy \right)^{-1} dr < \infty.$$

For all $r \geq 2y_0$, we have

$$\begin{aligned}
\int_0^r yN(x, y)dy &\geq \int_{y_0}^r \int_y^\infty yf(x, u)dudy \\
&= \int_{y_0}^r \int_y^r yf(x, u)dudy + \int_{y_0}^r \int_r^\infty yf(x, u)dudy \\
&\geq \int_{y_0}^r \int_y^r yf(x, u)dudy \\
&= \frac{r^3 - 3ry_0^2 + 2y_0^3}{6} f(x, r) \\
&\geq \frac{2y_0^3}{3} r^3 f(x, r).
\end{aligned}$$

Note that in the fourth inequality we took into account the fact that the densities $f(x, y)$ are decreasing on (y_0, ∞) for all $x \in \mathbb{R}$. Now, we have

$$\int_{2y_0}^\infty \left(\inf_{x \in \mathbb{R}} \int_0^r yN(x, y)dy \right)^{-1} dr \leq \frac{3}{2y_0^3} \int_{2y_0}^\infty \frac{dy}{y^3 \inf_{x \in \mathbb{R}} f(x, y)} \leq \frac{3}{2y_0^3} \int_{y_0}^\infty \frac{dy}{y^3 \inf_{x \in \mathbb{R}} f(x, y)},$$

which completes the proof. \square

Note that Theorem 3.11 can be strengthened. By assuming only uniform quasi-unimodality (and the condition in (3.2)) of the Lévy measure $\nu(x, dy)$, we have

$$\begin{aligned}
\int_r^\infty \frac{dy}{\inf_{x \in \mathbb{R}} \int_0^y u^2 \nu(x, du)} &\leq \int_r^\infty \frac{dy}{\inf_{x \in \mathbb{R}} \int_{y_0}^y u^2 f(x, u)du} \\
&\leq 3 \int_r^\infty \frac{d\xi}{(y^3 - y_0^3) \inf_{x \in \mathbb{R}} f(x, y)} \\
&\leq c \int_r^\infty \frac{d\xi}{y^3 \inf_{x \in \mathbb{R}} f(x, y)},
\end{aligned}$$

where $y_0 > 0$ is a constant of uniform quasi-unimodality of $\nu(x, dy)$ and $r > y_0$ and $c > \frac{3r^3}{r^3 - y_0^3}$ are arbitrary. Now, the claim is a direct consequence of Theorem 3.10. Let us also remark that in the Lévy process case the condition from Theorem 3.11 holds true even without the quasi-unimodality assumption, that is, the corresponding density does not have to be decreasing (see [22]).

We conclude this paper with some comparison conditions for the recurrence and transience in terms of the Lévy measure. Directly from Theorem 3.9 we can generalize the results from Theorem 3.5 and Corollary 3.6.

Theorem 3.12. *Let $\{F_t^1\}_{t \geq 0}$ and $\{F_t^2\}_{t \geq 0}$ be one-dimensional symmetric nice Feller processes with symbols $q_1(x, \xi)$ and $q_2(x, \xi)$ and Lévy measures $\nu_1(x, dy)$ and $\nu_2(x, dy)$, respectively. Let us put*

$$N_1(x, y) := \nu_1(x, (y, \infty)) \quad \text{and} \quad N_2(x, y) := \nu_2(x, (y, \infty)),$$

for $x \in \mathbb{R}$ and $y \geq 0$. If $N_1(x, y)$ has a bigger tail than $N_2(x, y)$, uniformly in $x \in \mathbb{R}$, then, for arbitrary $\rho > 0$,

$$\int_\rho^\infty \left(\sup_{x \in \mathbb{R}} \int_0^r yN_1(x, y)dy \right)^{-1} dr = \infty$$

implies

$$\int_{\rho}^{\infty} \left(\sup_{x \in \mathbb{R}} \int_0^r y N_2(x, y) dy \right)^{-1} dr = \infty.$$

In addition, if (3.12) holds true, then, for arbitrary $\rho > 0$,

$$\int_{\rho}^{\infty} \left(\inf_{x \in \mathbb{R}} \int_0^r y N_2(x, y) dy \right)^{-1} dr < \infty$$

implies

$$\int_{\rho}^{\infty} \left(\inf_{x \in \mathbb{R}} \int_0^r y N_1(x, y) dy \right)^{-1} dr < \infty.$$

Theorem 3.13. Let $\{F_t^1\}_{t \geq 0}$ and $\{F_t^2\}_{t \geq 0}$ be one-dimensional symmetric nice Feller processes with symbols $q_1(x, \xi)$ and $q_2(x, \xi)$ and Lévy measures $\nu_1(x, dy)$ and $\nu_2(x, dy)$, respectively. Further, assume that

- (i) $\nu_1(x, dy)$ is quasi-unimodal uniformly in $x \in \mathbb{R}$
- (ii) $\nu_1(x, dy)$ has a bigger tail than $\nu_2(x, dy)$ uniformly in $x \in \mathbb{R}$
- (iii) the function $x \mapsto \nu_1(x, O - x)$ is lower semicontinuous for every open set $O \subseteq \mathbb{R}$.

Then, for arbitrary $r > 0$,

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\sup_{x \in \mathbb{R}} q_1(x, \xi)} = \infty$$

implies

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\sup_{x \in \mathbb{R}} q_2(x, \xi)} = \infty.$$

In addition, if $q_1(x, \xi)$ satisfies (3.2) and (3.12), then, for all $r > 0$ small enough,

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\inf_{x \in \mathbb{R}} q_2(x, \xi)} < \infty$$

implies

$$\int_{\{|\xi| < r\}} \frac{d\xi}{\inf_{x \in \mathbb{R}} q_1(x, \xi)} < \infty.$$

Finally, let us remark that, according to Proposition 2.4, if the function $\xi \mapsto \inf_{x \in \mathbb{R}} \sqrt{q(x, \xi)}$ is subadditive, then the statements of Theorems 3.7, 3.9 and 3.13 and Corollary 3.8 (involving the conditions in (1.4) and (3.11)) do not depend on $r > 0$ and $\rho > 0$.

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